Summer School on Logic & Computation

University of Cape Town
6-9 November 2007

Willem L. Fouché (*Pretoria, South Africa*)

The Logic of the Smooth Topos

We give an informal description of the so-called smooth set of real numbers and how constructive logic enables us to work in this continuum. We discuss cartesian closed categories and explain how they provide us with models of universes containing the smooth set of real numbers. We conclude with a discussion of the implications of these ideas for physics.

Erich Grädel (*Aachen, Germany*)

Strategies and Algorithms for Infinite Games

Graph games, in which two or more players take turns (or not) to move a token through a directed graph, tracing out a possibly infinite path, have numerous applications, for instance for the design and verification of reactive systems, for the efficient evaluation of logical formulae, for planning and so on.

The fundamental mathematical questions on such games concern the existence of optimal strategies for the players, the complexity and structural properties of such strategies, and their realization by efficient algorithms. Which games are determined, in the sense that from each position, one of the players has a winning strategy? How to compute winning positions and optimal strategies? How much knowledge on the past of a play is necessary to determine an optimal next action? Which games are determined by memoryless strategies? And so on.

These questions are not only mathematically interesting, but translate into questions of design and verification of computing systems. The answers strongly depend on the particular form of a graph game, such as the number of players, the form of interaction between players (turn based or concurrent), the information that is available to the players, the structure of the game graph, and the type of the objectives or winning conditions of the players. A well-understood case are two-player, zero-sum games with perfect information and omega-regular objectives.

In this course, an overview will be given about fundamental concepts and results in the algorithmic theory of infinite games, about connections to other fields such as logic, descriptive set theory, automata theory, and verification, and about open problems and new challenges.

Joel David Hamkins (*New York NY, United States of America*)

Infinite Time Computation

We will give an introduction to the theory of infinite time Turing machines and related developments. Infinite time Turing machines extend the operation of ordinary Turing machines into transfinite ordinal time. By doing so, they provide a theoretical model of infinitary computability, while remaining close in spirit to many of the methods and concepts of classical computability. The model gives rise to a robust theory of infinitary computability on the reals, such as notions of computability for functions f:R to R and notions of decidability for subsets A of R, with a rich degree structure. Recent developments include the rise of infinitary complexity theory, with a solution of the infinite time analogue of the P vs. NP problem, the development of infinitary computable model theory and the introduction of several new models of ordinal computation, which strengthen connections with classical higher recursion theory. The topic lies on the ample boundaries between computability theory, descriptive set theory and constructible set theory. In this three-lecture course, we will give first a thorough introduction, and then explore a few of these latter topics. The course will be accessible to any logician familiar with basic concepts of computability and the transfinite ordinals.

Joseph S. Miller (*Storrs CT, United States of America*)

Effective Randomness

We will cover Kolmogorov complexity and Martin Löf randomness, especially as it related to computability theory. One theme will be results showing that the degree of randomness of a random infinite sequence is inversely proportional to its power as an oracle.

Valentin Goranko (*Johannesburg, South Africa*)

Finitely Presentable Infinite Structures

This course is an introduction to the investigation of the logical and algorithmic properties of infinite structures. These are becoming increasingly important in Computer Science with applications in, for instance, formal verification and constraint databases. Such structures necessarily have finite representations. These representations may be machine-theoretic, [for instance in terms of regular languages, or configurations graphs of abstract models of computation, such as Turing machines], or logical [for instance in terms of definability in some fixed effectively presentable structure]. Finite representations ensure that certain computational problems on that structure (usually expressible in some logic) are decidable.

Benedikt Löwe (*Amsterdam, The Netherlands*)

Determinacy Axioms and their Logical Strength

During this tutorial, we shall encounter game-theoretic axioms about infinite games, the so-called "determinacy axioms". The weakest of these axioms are provable, the stronger versions are part of the famous Gödel Incompleteness Phenomenon. We shall learn about the Mycielski-Steinhaus "Axiom of Determinacy" stating that all infinite two-player games admit a winning strategy for one player, see that it is incompatible with the axiom of choice, and see that its logical strength must be bigger than that of ordinary set theory.

Vasco Brattka (*Cape Town, South Africa*)

Computability on the Reals

Classical computability and complexity theory deals with discrete objects such as natural numbers and words. However, originally Turing invented his famous computational model to describe computations on real numbers. The theory built upon those early ideas is called computable analysis and the goal of this course is to give a brief introduction into this theory and some more recent developments.

The purpose of computable analysis is to handle computations with real numbers, subsets of the reals and functions on the reals in a physically realistic way that really reflects how real world computers can handle such objects. On the other hand, the theory should smoothly generalise concepts from classical computability theory and allow to express meaningful results.

Typical questions would be: in which sense can computers perform computations such as determination of zeros, calculation of derivatives and integrals? Which functions can computers actually compute? Which sets can physical computer actually plot?

Computable analysis allows to give conclusive answers to many such questions and the purpose of this course is to introduce the basic ideas and some of the required tools from the representation based approach to computable analysis.