Effective Rates of Convergence for Picard Iteration Sequences

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Introduction

- Let (X, d) be a metric space, let $f : X \to X$ and let $x_0 \in X$. The sequence $(f^n(x_0))$ is called the *Picard iteration sequence* with respect to f and x_0.
- We will here for the most part consider Picard iteration sequences, and we will often then write just (x_n).
- For particular classes of selfmaps f there are theorems stating convergence of all Picard iteration sequences, often to the unique fixed point of f.

The main points of this talk will be the following:

- We report on work in metric fixed point theory concerning a class of mappings called *asymptotic contractions*.
- We furthermore give a new application of a logical metatheorem due to U. Kohlenbach, explaining how one in certain cases can use techniques and insights from proof theory to find computable and highly uniform rates of convergence for the Picard iteration sequence for classes of not necessarily nonexpansive selfmaps of metric spaces.
- This will in a sense explain in logical terms why we were able to find explicit rates of convergence in two particular cases, namely for asymptotic contractions, and earlier for the so-called uniformly continuous *generalized p-contractive* mappings. However, for asymptotic contractions our rate of convergence holds in a setting more general than we can explain.

Proof mining: Extracting computational information from proofs

We take as starting point a logical metatheorem, due to Kohlenbach, with applications in metric fixed point theory. Based on Gödel's functional interpretation.

Basic reference:

- U. Kohlenbach. Some logical metatheorems with applications in functional analysis. *Transactions of the American Mathematical Society*, 357:89-128, 2005.

Developed further in:

- P. Gerhardy and U. Kohlenbach. General logical metatheorems for functional analysis. To appear in *Transactions of the American Mathematical Society*.

The setting

- We use $\mathcal{A}^\omega$, a formal system for analysis. Basically Peano arithmetic in all finite types with quantifier-free axiom of choice, dependent choice and countable choice, but with a certain quantifier-free rule of extensionality instead of the full axiom of extensionality in all types.
- Higher type equality is defined extensionally and real numbers are represented as type 1 objects.
- $\mathcal{A}^\omega[X, d]$ is the theory for which the relevant metatheorem is proved. An extension of $\mathcal{A}^\omega$ to new types. We "add" an abstract bounded metric space. Among other things we have a new ground type X and a new constant d_X of type $X \to X \to 1$. $\mathbf{T}^X$ is the set of all finite types over the two ground types 0 and X, i.e.

$$0, X \in \mathbf{T}^X, \text{ and if } \rho, \tau \in \mathbf{T}^X, \text{ then } \rho \to \tau \in \mathbf{T}^X.$$

Definition 1. (Howard, Bezem, Kohlenbach) *The relation* x^* s-maj$_\rho$ x *("strong majorizability") between functionals of type* $\rho \in \mathbf{T}^X$ *is defined by induction on the type as follows.*

$$m \text{ s-maj}_0\ n :\equiv m \geq_0 n \wedge m, n \in \mathbb{N},$$

$$x^* \text{ s-maj}_X\ x :\equiv x^*, x \in X,$$

$$x^* \text{ s-maj}_{\sigma\to\tau}\ x :\equiv \forall y^*, y(y^* \text{ s-maj}_\sigma\ y \to x^*(y^*) \text{ s-maj}_\tau\ x(y) \wedge x^*(y^*) \text{ s-maj}_\tau\ x^*(y)).$$

Definition 2. *A type* ρ *has degree*

- 1 *if* $\rho = 0 \to \ldots \to 0$ *(including* $\rho = 0$*).*
- 2 *if* $\rho = \tau_1 \to \ldots \to \tau_k \to 0$ *(including* $\rho = 0$*), with* τ_i *of degree* 1.
- $(0, X)$ *if* $\rho = 0 \to \ldots \to 0 \to X$ *(including* $\rho = X$*).*
- $(1, X)$ *if* $\rho = \tau_1 \to \ldots \to \tau_k \to X$ *(including* $\rho = X$*), where* τ_i *has degree* 1 *or* $(0, X)$*.*

- Let $\mathcal{A}^\omega[X,d] + \Delta$ be $\mathcal{A}^\omega[X,d]$ extended with a constant f of type $X \to X$, with other constants $c_1, \ldots, c_m$ of type of degree 2 or $(1, X)$, and with purely universal axioms (which do not contain $\vee$, and with the types of all quantifiers of degree 2 or $(1, X)$).

- A consequence of the metatheorem is that if

$$\mathcal{A}^\omega[X,d] + \Delta \vdash \forall k^0 \forall x_0^X, y_0^X \exists n^0 (d_X(x_n, y_n) <_{\mathbb{R}} 2^{-k}),$$

then from the proof we can find a computable functional $\Phi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that

$$\forall k \in \mathbb{N} \forall x, y \in X \exists n \leq \Phi(k, b)(d(f^n(x), f^n(y)) < 2^{-k})$$

holds in all nonempty b-bounded metric spaces (X, d) and for all $f : X \to X$ such that X and f satisfy the purely universal axioms added to $\mathcal{A}^\omega[X,d]$ (under a suitable interpretation of the new constants).

- Φ is dependent on majorants c_i^* of those c_i which are of type of degree 2.

- If we also know that $f : X \to X$ has a fixed point $z \in X$, then Φ yields a *rate of proximity* which is uniform in x:

$$\forall k \in \mathbb{N} \forall x \in X \exists n \leq \Phi(k, b)(d(f^n(x), z) < 2^{-k}).$$

Definition 3. *Let (X, d) be a metric space, and let (x_n) be a sequence in X. Let $z \in X$. We say that $\phi : \mathbb{N} \to \mathbb{N}$ is a* rate of proximity *for (x_n) to z if ϕ satisfies*

$$\forall k \in \mathbb{N} \exists n \leq \phi(k)(d(x_n, z) < 2^{-k}).$$

- A mapping $f : X \to X$ is *nonexpansive* if

$$\forall x, y \in X (d(f(x), f(y)) \leq d(x, y)).$$

If the mapping f is nonexpansive, then we can conclude that a rate of proximity is also a rate of convergence.

If the mapping f is not nonexpansive, then we cannot easily conclude that a rate of proximity is also a rate of convergence.

- We can, however, give sufficiency criteria for when we can find computable rates of convergence also in this case. This covers to a large extent the two concrete cases where we have found explicit rates of convergence.
- Although checking whether the criteria are fulfilled is nontrivial this nonetheless constitutes a possible method for how to go about finding rates of convergence without assuming the mappings to be nonexpansive.

Rates of convergence without nonexpansivity

For $m \in \mathbb{N}$, $m \geq 1$, we can consider the product space (X^m, d_m) with $d_m(\vec{x}, \vec{y}) = \max\{d(x^1, y^1), \ldots, d(x^m, y^m)\}$ for $\vec{x} = (x^1, \ldots, x^m)$, $\vec{y} = (y^1, \ldots, y^m)$, and with $f_m : X^m \to X^m$ defined by $f_m(\vec{x}) = (f(x^1), \ldots, f(x^m))$.

Theorem. Assume that for the (X,d) and $f: X \to X$ axiomatized in $\mathcal{A}^\omega[X,d] + \Delta$ and for each $m \geq 1$ we have that (X^m, d_m) and $f_m : X^m \to X^m$ satisfy the purely universal axioms added to $\mathcal{A}^\omega[X,d]$, with each of the new constants c_i added to $\mathcal{L}(\mathcal{A}^\omega[X,d])$ interpreted by functionals which for each constant have a *common* majorizer for all $m \geq 1$. Assume further that

$$\mathcal{A}^\omega[X,d] + \Delta \vdash \forall k^0 \forall x_0^X, y_0^X \exists n^0 (d_X(x_n, y_n) <_{\mathbb{R}} 2^{-k}).$$

Then from the proof one can find a computable functional $\Phi' : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ s.t.

$$\forall k \in \mathbb{N} \forall x, y \in X \forall n \geq \Phi'(k,b) (d(f^n(x), f^n(y)) < 2^{-k})$$

holds for all nonempty b-bounded metric spaces (X,d) and all $f : X \to X$ s.t. X and f satisfy the purely universal axioms added to $\mathcal{A}^\omega[X,d]$ (under a suitable interpretation of the new constants).

A case study: Asymptotic contractions

- Asymptotic contractions were introduced by W.A. Kirk in 2003. (In: *Fixed points of asymptotic contractions.* J. Math. Anal. Appl., 277:645-650, 2003.)

Definition 4. (Kirk) *A function* $f : X \to X$ *on a metric space* (X, d) *is called an* asymptotic contraction in the sense of Kirk *with moduli* $\phi, \phi_n : [0, \infty) \to [0, \infty)$ *if* ϕ, ϕ_n *are continuous,* $\phi(s) < s$ *for all* $s > 0$ *and for all* $x, y \in X$

$$d(f^n(x), f^n(y)) \leq \phi_n(d(x, y)),$$

and moreover $\phi_n \to \phi$ *uniformly on the range of* d.

- Asymptotic contractions are not necessarily nonexpansive.

Kirk's fixed point theorem for asymptotic contractions

Theorem 5. (Kirk) *Let (X, d) be a complete metric space, and let $f : X \to X$ be a continuous asymptotic contraction in the sense of Kirk. If for some $x_0 \in X$ the Picard iteration sequence (x_n) is bounded, then f has a unique fixed point $z \in X$ and for every starting point $x_0 \in X$ the iteration sequence (x_n) converges to z.*

This has given rise to several papers dealing with versions of asymptotic contractions:

- I.D. Arandelović. *On a fixed point theorem of Kirk.* J. Math. Anal. Appl., 301:384-385, 2005.
- M. Arav, F.E. Castillo Santos, S. Reich, A.J. Zaslavski. *A note on asymptotic contractions.* Fixed Point Theory Appl., vol. 2007, Article ID 39465, 6 pages, 2007, doi:10.1155/2007/39465.

- E.M. Briseid. *A rate of convergence for asymptotic contractions.* J. Math. Anal. Appl., 330:364-376, 2007.
- E.M. Briseid. *Some results on Kirk's asymptotic contractions.* Fixed Point Theory, Volume 8, No. 1, 2007, 17-27.
- E.M. Briseid. *Addendum to the paper: Some results on Kirk's asymptotic contractions.* To appear in Fixed Point Theory.
- Y.-Z. Chen. *Asymptotic fixed points for nonlinear contractions.* Fixed Point Theory Appl., 2005:2, pages 213-217, 2005.
- P. Gerhardy. *A quantitative version of Kirk's fixed point theorem for asymptotic contractions.* J. Math. Anal. Appl., 316:339-345, 2006.
- J. Jachymski and I. Jóźwik. *On Kirk's asymptotic contractions.* J. Math. Anal. Appl., 300:147-159, 2004.

- T. Suzuki. *Fixed-point theorem for asymptotic contractions of Meir-Keeler type in complete metric spaces.* Nonlinear Analysis, 64:971-978, 2006.
- T. Suzuki. *A definitive result on asymptotic contractions.* J. Math. Anal. Appl., 335:707-715, 2007.
- K. Włodarczyk, D. Klim, R. Plebaniak. *Existence and uniqueness of endpoints of closed set-valued asymptotic contractions in metric spaces.* J. Math. Anal. Appl., 328:46-57, 2007.
- H.-K. Xu. *Asymptotic and weakly asymptotic contractions.* Indian Journal of Pure and Applied Mathematics, 36(3):145-150, 2005.

Description of main results on asymptotic contractions

- Our results build on the analysis of Kirk's fixed point theorem for asymptotic contractions given by Gerhardy. He uses techniques and insights from proof mining to give an elementary proof of Kirk's theorem, and also a rate of proximity to the fixed point.
- We give an explicit rate of convergence to the fixed point for any sequence $(f^n(x))_n$, without assuming that f is nonexpansive. This amounts to a fully effective version of Kirk's theorem on asymptotic contractions with an elementary proof.
- This rate of convergence depends on the space, the mapping and the starting point through a bound on the iteration sequence and some moduli for the mapping appearing as parameters, but is otherwise fully uniform.

- A weaker result is guaranteed by our novel application of the logical metatheorem due to Kohlenbach, namely that a *uniformly continuous* asymptotic contraction on a *bounded* metric space has a computable rate of convergence which instead depends on a bound on the space, a modulus of uniform continuity for the mapping and some other moduli. Something similar could earlier only be guaranteed by Kohlenbach's theorem in the case where f is also nonexpansive. (Through a rate of proximity.)

As a byproduct of the uniformity features of the analysis we get also the following:

- We give a characterization of asymptotic contractions in the sense of Kirk on nonempty, bounded, complete metric spaces, finding that they are exactly the mappings for which every Picard iteration sequence converges to the same point with a rate of convergence which is uniform in the starting point.

As an essential part of his analysis Gerhardy modifies the definition of an asymptotic contraction given by Kirk, and then shows that the new definition covers the old one. We use a slightly generalized version.

Remark 6. *An asymptotic contraction in the sense of Kirk is also an asymptotic contraction in the sense of the definition below. We will often drop the superscripts from the* η^b, β^b *appearing in this definition, and we will call the mappings defined just asymptotic contractions.*

Definition 7. *A function* $f : X \to X$ *on a metric space* (X, d) *is called a* (generalized) asymptotic contraction *if for each* $b > 0$ *there exist moduli* $\eta^b : (0, b] \to (0, 1)$ *and* $\beta^b : (0, b] \times (0, \infty) \to \mathbb{N}$ *and the following hold:*

1. *There exists a sequence of functions* $\phi_n^b : (0,\infty) \to (0,\infty)$ *such that for each* $0 < l \leq b$ *the function* $\beta_l^b := \beta^b(l,\cdot)$ *is a modulus of uniform convergence for* $(\phi_n^b)_{n\in\mathbb{N}}$ *on* $[l,b]$, *i.e.*

   $$\forall \varepsilon > 0 \forall s \in [l,b] \forall m,n \geq \beta_l^b(\varepsilon)(|\phi_m^b(s) - \phi_n^b(s)| \leq \varepsilon).$$

   Furthermore, if $\varepsilon < \varepsilon'$ *then* $\beta_l^b(\varepsilon) \geq \beta_l^b(\varepsilon')$.

2. *For all* $x, y \in X$, *for all* $\varepsilon > 0$ *and for all* $n \in \mathbb{N}$ *such that* $\beta_\varepsilon^b(1) \leq n$, *we have:*

   $$b \geq d(x,y) \geq \varepsilon \quad \textit{gives} \quad d(f^n(x), f^n(y)) \leq \phi_n^b(\varepsilon) d(x,y).$$

3. *Define* $\phi^b := \lim_{n\to\infty} \phi_n^b$. *Then for each* $0 < \varepsilon \leq b$ *we have*

   $$\phi^b(s) + \eta^b(\varepsilon) \leq 1$$

   for each $s \in [\varepsilon, b]$.

Main results on asymptotic contractions

Theorem 8. (In: *A rate of convergence for asymptotic contractions.* J. Math. Anal. Appl., 330:364-376, 2007.)

Let (X, d) be a nonempty complete metric space, and let $f : X \to X$ be a continuous asymptotic contraction with moduli η^b and β^b for each $b > 0$. Then for all $x_0 \in X$ the sequence (x_n) converges to the unique fixed point $z \in X$ of f. If for $x_0 \in X$ the sequence (x_n) is bounded by $b > 0$, then (x_n) has the following rate of convergence. Let $b \geq \varepsilon > 0$ and let $n \in \mathbb{N}$ satisfy

$$n \geq \max\{k \cdot (2M_\gamma + \beta_{(\frac{\varepsilon}{2})}(\delta) + K_\gamma - 1), \\ (k-1) \cdot (2M_\gamma + \beta_{(\frac{\varepsilon}{2})}(\delta) + K_\gamma - 1) + M_\gamma + 1\},$$

where $k := \left\lceil \frac{\lg(\varepsilon) - \lg(b)}{\lg(1 - \frac{\eta(\gamma)}{2})} \right\rceil$, $M_\gamma := K_\gamma \cdot \left\lceil \frac{\lg(\gamma) - \lg(b)}{\lg(1 - \frac{\eta(\gamma)}{2})} \right\rceil$, $K_\gamma := \beta_\gamma(\frac{\eta(\gamma)}{2})$, *and* $\delta := \min\{\frac{\varepsilon}{2}, \frac{\eta(\frac{\varepsilon}{2})}{2}\}$, $\gamma := \min\{\delta, \frac{\delta\varepsilon}{4}\}$. *Then* $d(x_n, z) \leq \varepsilon$.

- We remark that Theorem 8 provides a rate of convergence which given η, β and b is *uniform* in the function f, the space and the starting point.
- Kirk's original theorem follows from Theorem 8.
- Without continuity we get the same rate of convergence to a common limit, which then might not be a fixed point.
- More recently we have constructed a rate of convergence which does not depend on a bound b on the iteration sequence, but which instead depends on (strictly positive) upper and lower bounds b_1, b_2 on $d(x_0, f(x_0))$, i.e. such that $0 < b_2 \leq d(x_0, f(x_0)) \leq b_1$. (In: *Some results on Kirk's asymptotic contractions.* Fixed Point Theory, Volume 8, No. 1, 2007, 17-27.)

Theorem 9. (In: *A rate of convergence for asymptotic contractions.* J. Math. Anal. Appl., 330:364-376, 2007.)

Let (X, d) *be a nonempty, bounded, complete metric space, and let* $f : X \to X$. *Then the following are equivalent:*

1. *The function* f *is an asymptotic contraction.*
2. *The function* f *is an asymptotic contraction in the sense of Kirk.*
3. *There exists* $z \in X$ *such that for each* $x_0 \in X$ *the Picard iteration sequence converges to* z *with a rate of convergence which is uniform in the starting point.*
4. *There exists* $\alpha : (0, \infty) \to \mathbb{N}$ *such that for all* $x, y \in X$,

$$\forall \varepsilon > 0 \forall n \geq \alpha(\varepsilon) \left(d(x, y) \geq \varepsilon \to d(f^n(x), f^n(y)) \leq \frac{1}{2} d(x, y) \right).$$