Complexity of Δ^1_2 sets and μ-Calculus: when Infinite Games Make Modal Logic and Descriptive Set Theory Meet

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What is μ-calculus?

propositional modal μ-calculus

$$=$$

$$\left\{ \begin{array}{c} \text{propositional modal logic} \\ + \\ \text{least and greatest fixpoint operators} \end{array} \right.$$

Why the μ in μ-calculus ?

φ a monotone function from a complete ordered set into itself.

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$$\mu x.\varphi(x) = \text{ the least fixpoint of } \varphi$$

$$\nu x.\varphi(x) = \text{ the greatest fixpoint of } \varphi$$

Why the μ in μ-calculus ?

Take φ a modal formula, x occurs **only positively** in φ.

Let $\mathbf{K} = \langle W, R \rangle$ be any Kripke structure (any transition system), and $V : P \rightarrow \wp(W)$ any valuation.

Why the μ in μ-calculus ?

Take φ a modal formula, x occurs **only positively** in φ.

Let $\mathbf{K} = \langle W, R \rangle$ be any Kripke structure (any transition system), and $V : P \rightarrow \wp(W)$ any valuation.

$$\begin{aligned} \varphi(x) : \qquad & \wp(W) \rightarrow \wp(W) \\ & ||x|| = S \mapsto ||\varphi(x)||_{x \mapsto S} \end{aligned}$$

is a **monotone function** from $\langle \wp(K), \subseteq \rangle$ into itself.

Syntax

$$\varphi ::= \underbrace{p \mid \top \mid \bot \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi}_{\text{propositional logic}}$$

Syntax

$$\varphi ::= \underbrace{p \mid \top \mid \bot \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Diamond\varphi \mid \Box\varphi}_{\text{modal logic}}$$

Syntax

$$\varphi ::= \underbrace{p \mid \top \mid \bot \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Diamond\varphi \mid \Box\varphi \mid \mu x.\varphi \mid \nu x.\varphi}_{\mu\text{-calculus}}$$

where x is a propositional variable which **occurs only positively** in $\eta x.\varphi$ ($\eta = \nu, \mu$).

Why μ-calculus?

The fixpoint operators makes the calculus very powerful (and really hard to understand....)

Why μ-calculus?

It can express :

liveness: “something good (p) will happen”

$$\mu x.p \vee \Diamond x$$

Why μ-calculus?

It can express :

safety: "something bad (p) will never happen"

$$\nu x.\neg p \wedge \Box x$$

Why μ-calculus?

It can express :

fairness: "something good (p) will happen infinitely often"

$$\nu x.\mu y.(p \wedge \Diamond x) \vee \Diamond y$$

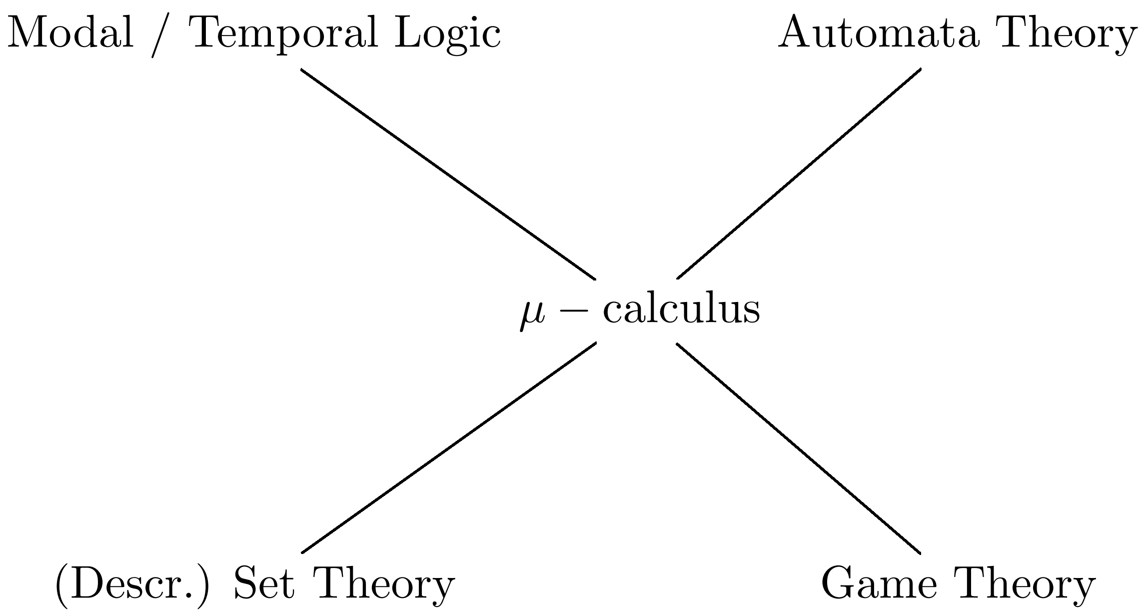
Why μ-calculus?

and much more...

Walukiewicz formula W_n:

$$\mu x_n \nu x_{n-1} \ldots \mu x_1 \Big((p \rightarrow \Diamond \bigwedge_{1 \leq n} (r \rightarrow x_i)) \wedge (\neg p \rightarrow \Box \bigwedge_{1 \leq n} (r \rightarrow x_i)) \Big)$$

Why μ-calculus?



Semantics

Let $\mathbf{K} = \langle W, R\rangle$ be a Kripke structure (a transition system) and $V : P \to \wp(W)$ a valuation.

$$||p||^{\mathbf{K}}_{V} = V(p)$$

$$||\top||^{\mathbf{K}}_{V} = W$$

$$||\bot||^{\mathbf{K}}_{V} = \emptyset$$

$$||\neg\varphi||^{\mathbf{K}}_{V} = W \setminus ||\varphi||^{\mathbf{K}}_{V}$$

$$||\varphi \wedge \psi||^{\mathbf{K}}_{V} = ||\varphi||^{\mathbf{K}}_{V} \cap ||\psi||^{\mathbf{K}}_{V}$$

$$||\varphi \vee \psi||^{\mathbf{K}}_{V} = ||\varphi||^{\mathbf{K}}_{V} \cup ||\psi||^{\mathbf{K}}_{V}$$

Semantics

Let $\mathbf{K} = \langle W, R \rangle$ be a Kripke structure (a transition system) and $V : P \to \wp(W)$ a valuation.

$$||\Diamond\varphi||_V^{\mathbf{K}} = \{s \in W : \exists t.\langle s, t\rangle \in R \wedge t \in ||\varphi||_V^{\mathbf{K}}\}$$

$$||\Box\varphi||_V^{\mathbf{K}} = \{s \in W : \forall t.(\langle s, t\rangle \in R) \to (t \in ||\varphi||_V^{\mathbf{K}})\}$$

Semantics

Let $\mathbf{K} = \langle W, R \rangle$ be a Kripke structure (a transition system) and $V : P \to \wp(W)$ a valuation.

$$||\mu x.\varphi(x)||_V^{\mathbf{K}} = \bigcap \{S \subseteq W : ||\varphi(S)||_V^{\mathbf{K}} \subseteq S\}$$

$$||\nu x.\varphi(x)||_V^{\mathbf{K}} = \bigcup \{S \subseteq W : S \subseteq ||\varphi(S)||_V^{\mathbf{K}}\}$$

Semantics

$$||\varphi|| = \{\langle \mathbf{K}, V, s\rangle : s \in W \wedge s \in ||\varphi||^{\mathbf{K}}_{V}\}$$

Completeness

Kozen's axioms for the μ-calculus:

- axioms and rules for the modal logic K
- the "unfolding" axioms: $\varphi(\mu x.\varphi(x)) \to \mu x.\varphi(x)$,
- Park's fixpoint induction rule: $\langle \varphi(\psi) \to \psi, \mu x.\varphi(x) \to \psi \rangle$

Completeness

Theorem 1 (Walukiewicz (93)) *The μ-calculus has a complete and sound axiomatization.*

From now on. . .

From now on. . .

. . . forget what you have seen about the semantics.

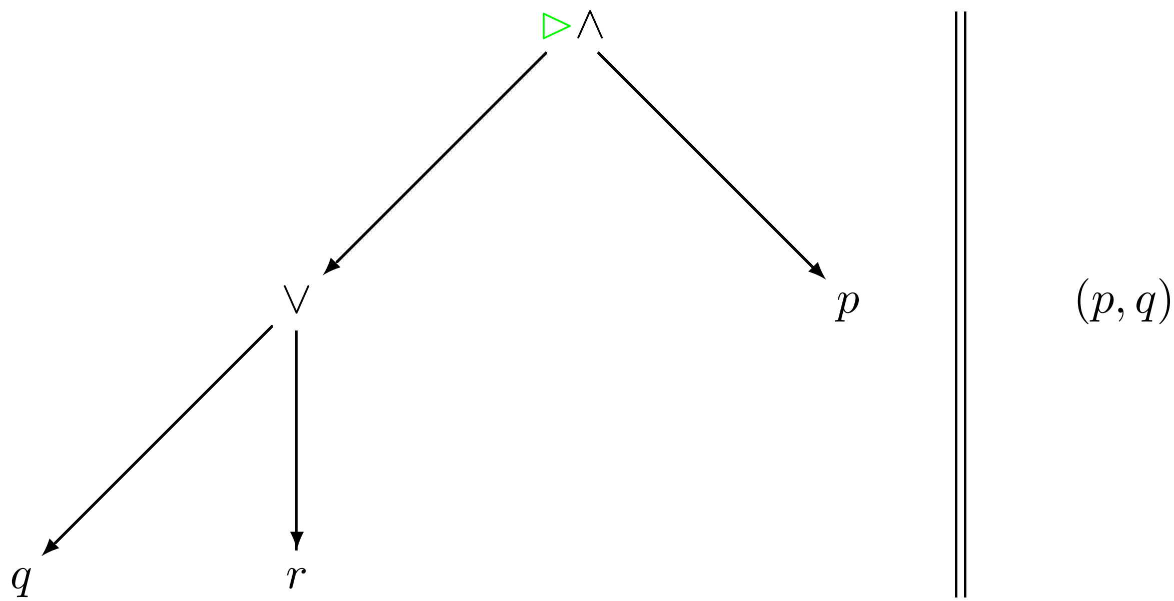
Let's play games.

Evaluation game for classical propositional logic

position	player	next position
$\langle p_i, w\rangle$	-	-
$\langle \neg\psi, w\rangle$	**V** and **F** switch their roles	$\langle \psi, w\rangle$
$\langle \psi \vee \phi, w\rangle$	**V** chooses between $\langle \psi, w\rangle$ and $\langle \phi, w\rangle$	**V** choice
$\langle \psi \wedge \phi, w\rangle$	**F** chooses between $\langle \psi, w\rangle$ and $\langle \phi, w\rangle$	**F** choice

Example 1

$$(q \vee r) \wedge p$$

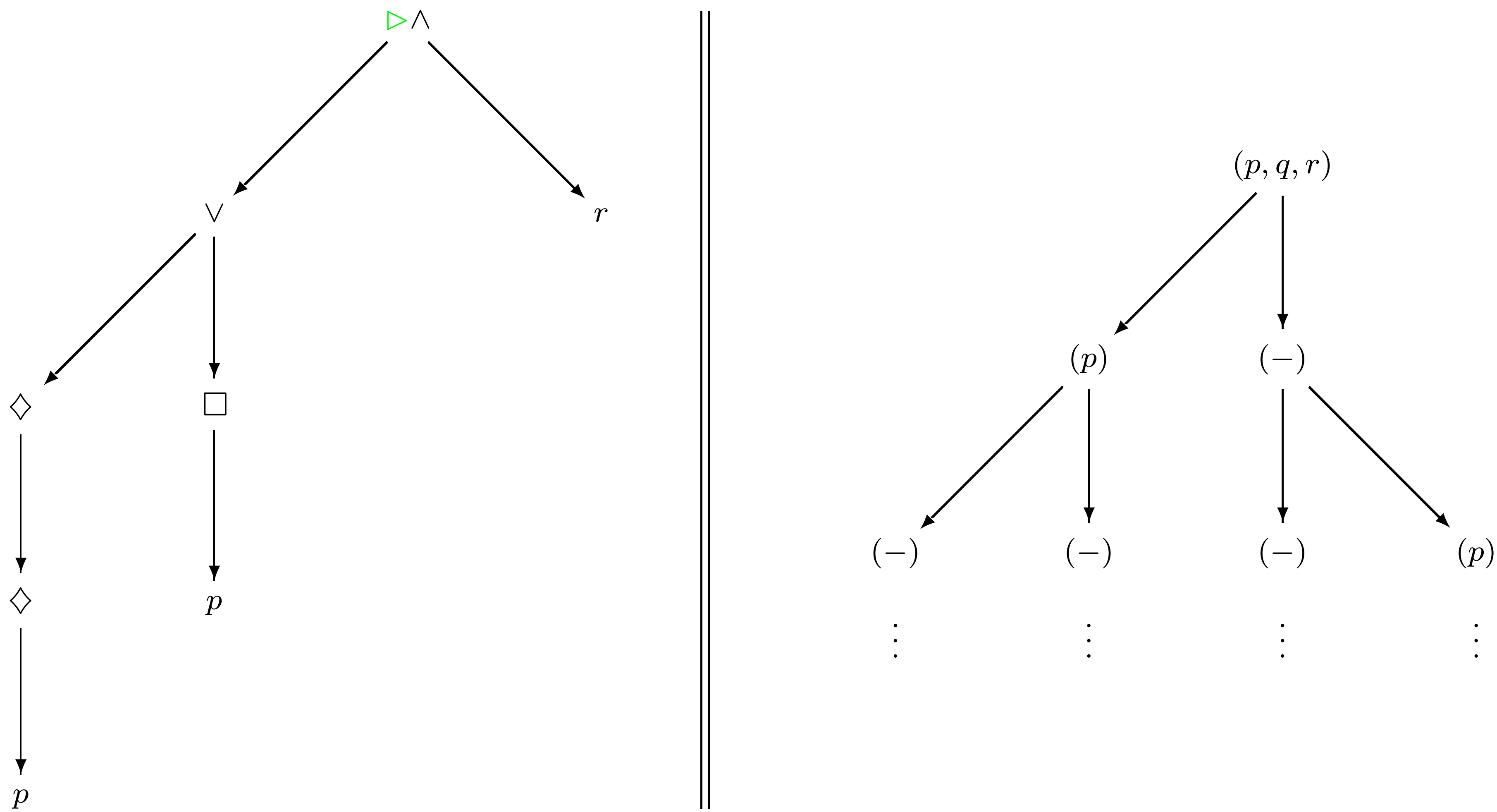


Evaluation game for modal logic

position	player	next position
$\langle p_i, w\rangle$	-	-
$\langle \neg\psi, w\rangle$	**V** and **F** switch their roles	$\langle \psi, w\rangle$
$\langle \psi \vee \phi, w\rangle$	**V** chooses between $\langle \psi, w\rangle$ and $\langle \phi, w\rangle$	**V** choice
$\langle \psi \wedge \phi, w\rangle$	**F** chooses between $\langle \psi, w\rangle$ and $\langle \phi, w\rangle$	**F** choice
$\langle \Diamond\psi, w\rangle$	**V** chooses a node w' s.t. $w \longrightarrow w'$	$\langle \psi, w'\rangle$
$\langle \Box\psi, w\rangle$	**F** chooses a node w' s.t. $w \longrightarrow w'$	$\langle \psi, w'\rangle$

Example 2

$$(\Diamond\Diamond p \vee \Box p) \wedge r$$



Evaluation game for μ-calculus

position	player	next position
$\langle p_i, w \rangle$	-	-
$\langle \neg\psi, w \rangle$	**V** and **F** switch their roles	$\langle \psi, w \rangle$
$\langle \psi \vee \phi, w \rangle$	**V** chooses between $\langle \psi, w \rangle$ and $\langle \phi, w \rangle$	**V** choice
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$\langle \Diamond\psi, w \rangle$	**V** chooses a node w' s.t. $w \longrightarrow w'$	$\langle \psi, w' \rangle$
$\langle \Box\psi, w \rangle$	**F** chooses a node w' s.t. $w \longrightarrow w'$	$\langle \psi, w' \rangle$
$\langle \mu x.\psi, w \rangle$	-	$\langle \psi, w \rangle$
$\langle \nu x.\psi, w \rangle$	-	$\langle \psi, w \rangle$
$\langle x, w \rangle$	-	$\langle \psi_x, w \rangle$

From a game-theoretical point of view...

...the game behind the μ-calculus is a **parity game**

From a game-theoretical point of view...

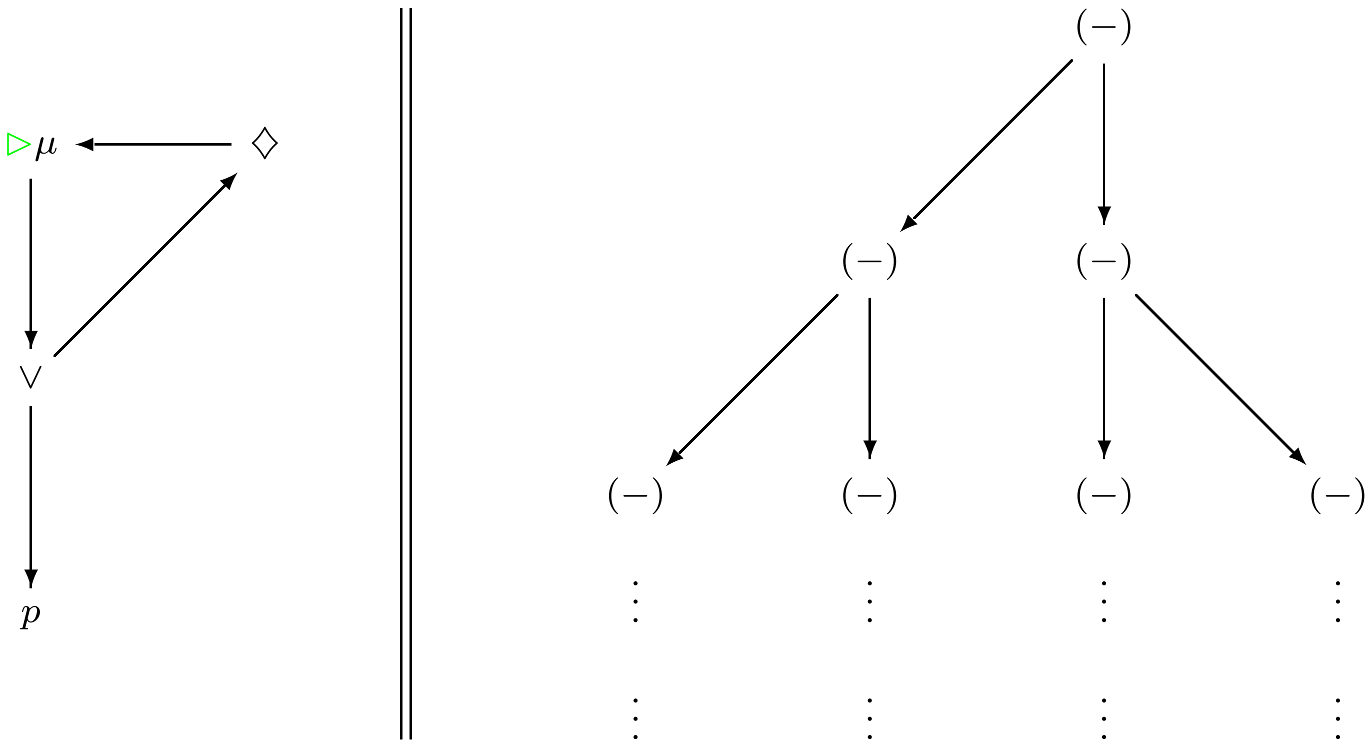
... "μ means **finite** looping"

From a game-theoretical point of view...

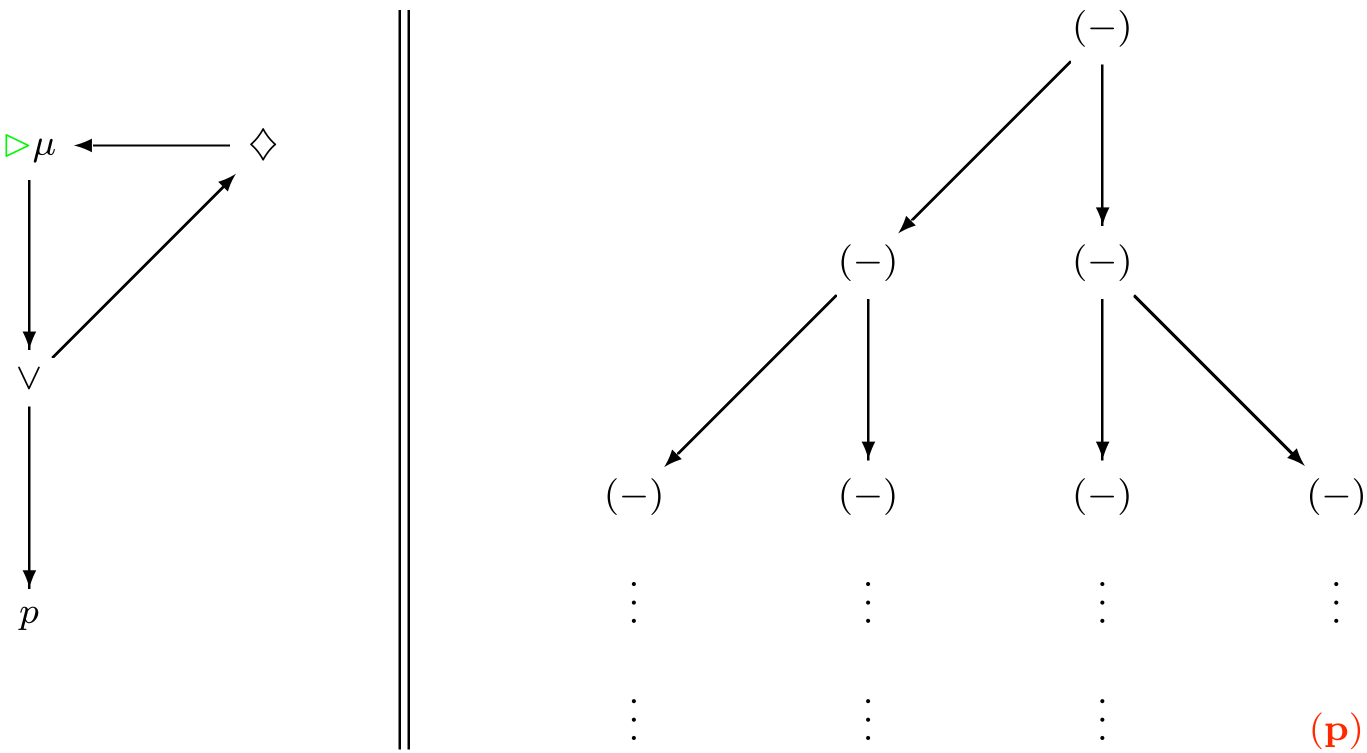
... "ν means **infinite** looping"

Example 3

$$\mu x(p \vee \Diamond x)$$

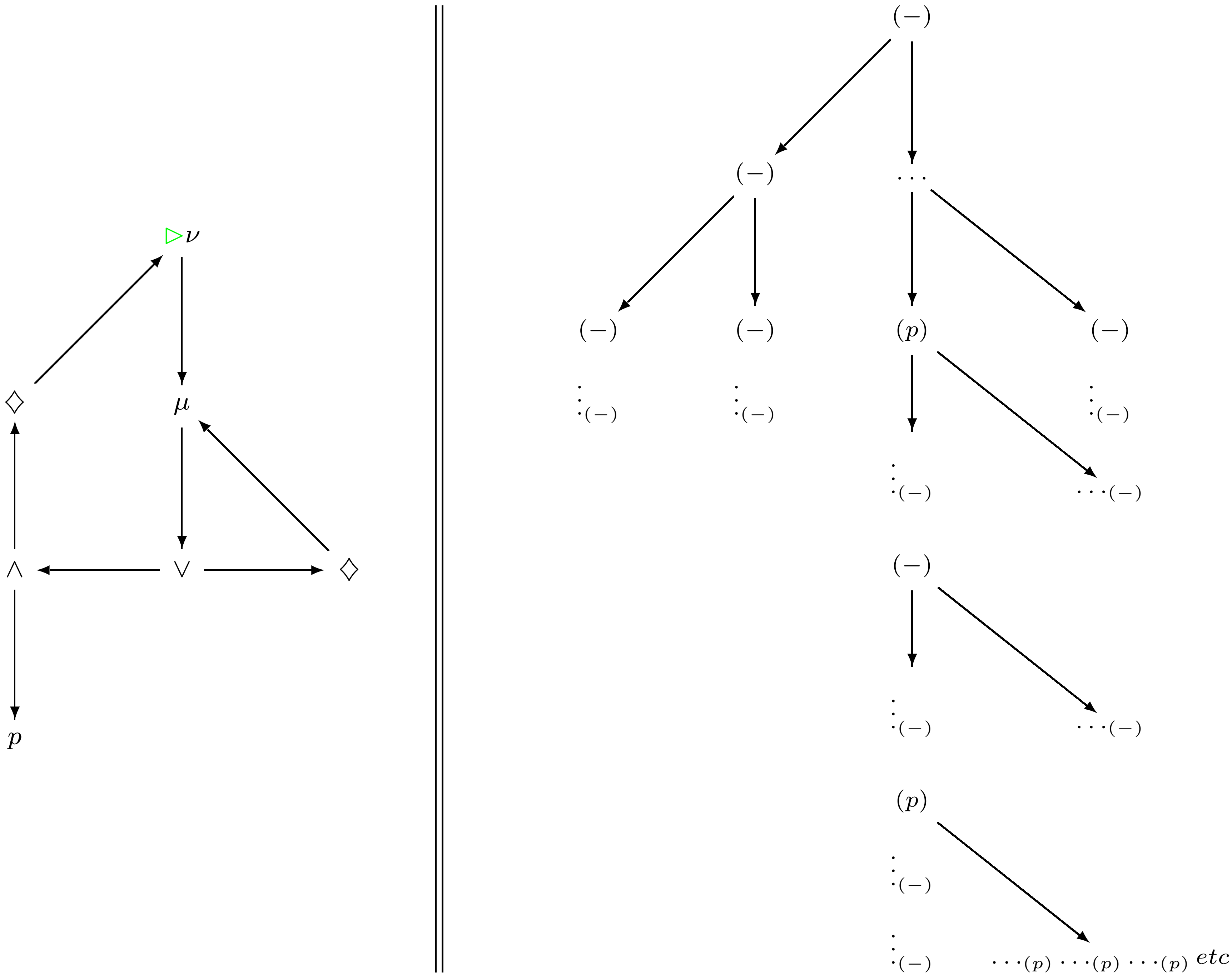


$$\mu x(p \vee \Diamond x)$$



Example 4

$$\nu x \Big(\mu y ((p \wedge \Diamond x) \vee \Diamond y) \Big)$$



▷ν
μ
◇
∧
∨
◇
p
(−)
(p)
etc

Proposition 1 (Game-theoretical version of the "fundamental theorem") $s \in ||\varphi||_V^{\mathbf{K}}$ *iff* $\boldsymbol{V}$ *has a winning strategy in* $\mathcal{E}(\varphi, \langle \mathbf{K}, V \rangle)$ *starting at* s*.*

The fixpoint alternation depth

The fixpoint alternation depth (fad) of a formula is the number of non-trivial nestings of alternating least and greatest fixpoints.

The fixpoint alternation depth

- $$\varphi_1 := (p \vee \Diamond q) \wedge \Box r$$

$$fad(\varphi_1) = 0$$

The fixpoint alternation depth

- $$\varphi_2 := \mu x(p \vee \Diamond x) \wedge \Box r$$

The fixpoint alternation depth

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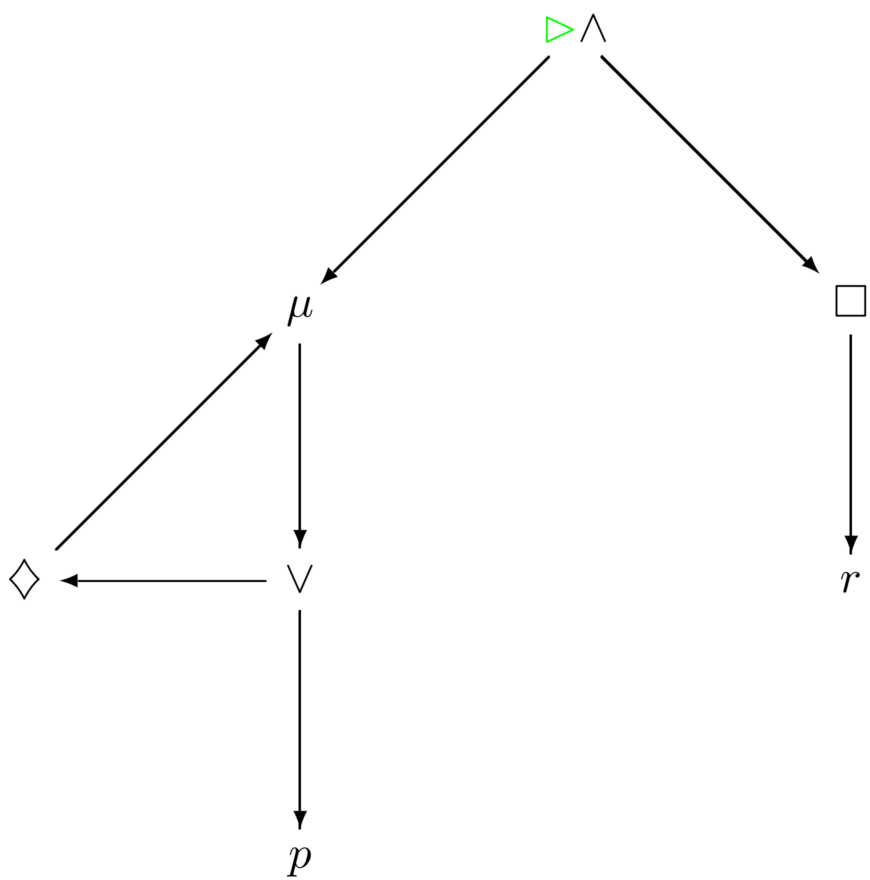


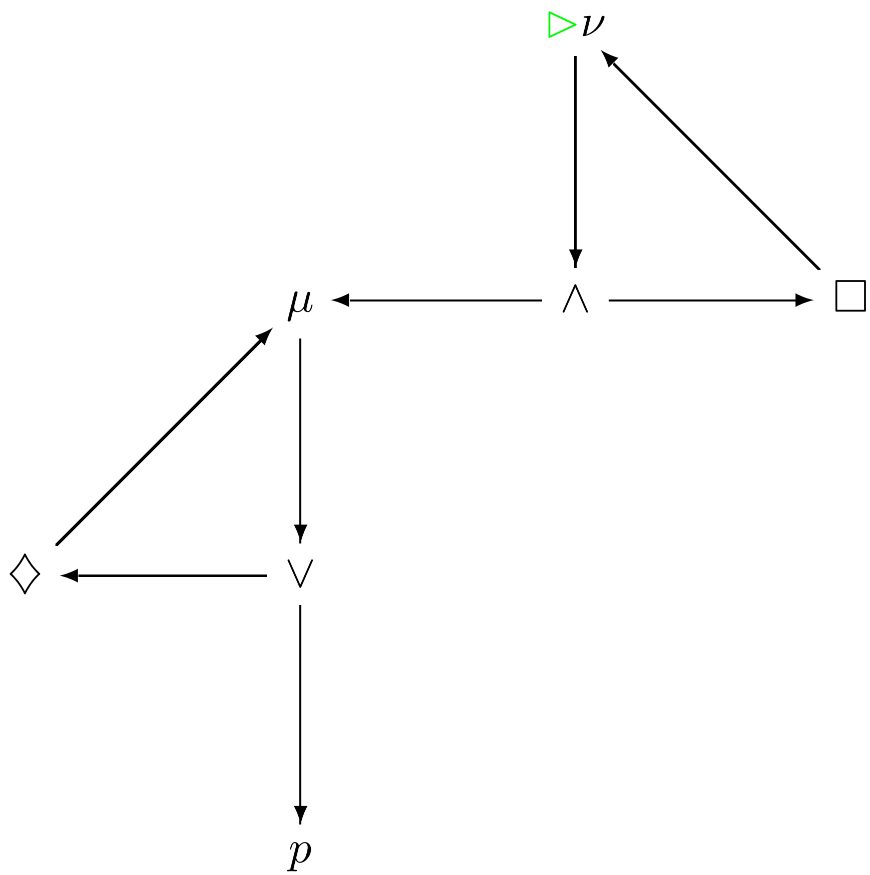
The fixpoint alternation depth

- $$fad(\varphi_2) = 1$$

The fixpoint alternation depth

- $$\varphi_3 := \nu y\big(\mu x(p \vee \Diamond x) \wedge \Box y\big)$$

The fixpoint alternation depth

- 

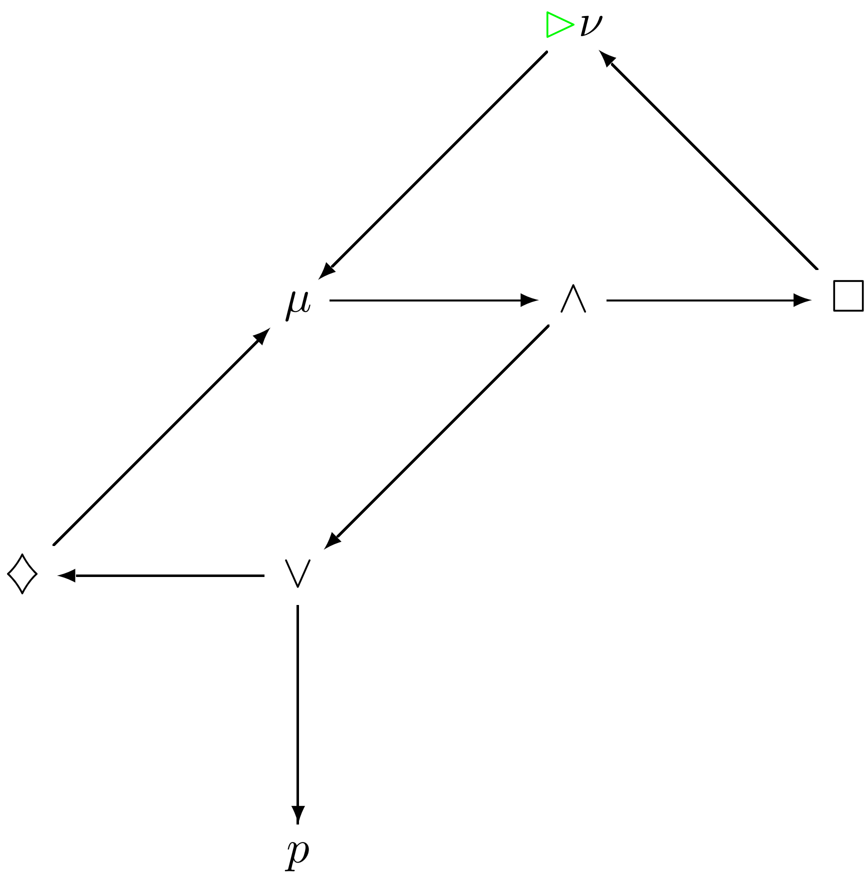
The fixpoint alternation depth

- $$fad(\varphi_3) = 1$$

The fixpoint alternation depth

- $$\varphi_4 := \nu y\big(\mu x((p \vee \Diamond x) \wedge \Box y)\big)$$

The fixpoint alternation depth

- 

The fixpoint alternation depth

- $$fad(\varphi_4) = 2$$

The syntactical fixpoint alternation hierarchy

For $\eta = \nu, \mu$, $\eta(\Phi)$ is the smallest class of formulae s.t.:

- $\Phi, \neg\Phi \subset \eta(\Phi)$;
- If $\psi(x) \in \eta(\Phi)$ and x occurs only positively, then $\eta x.\psi \in \eta(\Phi)$;
- If $\psi, \varphi \in \eta(\Phi)$, then $\psi \wedge \varphi, \psi \vee \varphi, \Diamond\psi, \Box\psi \in \eta(\Phi)$;
- If $\psi, \varphi \in \eta(\Phi)$ and x is bound in ψ, then $\varphi[\psi / x] \in \eta(\Phi)$

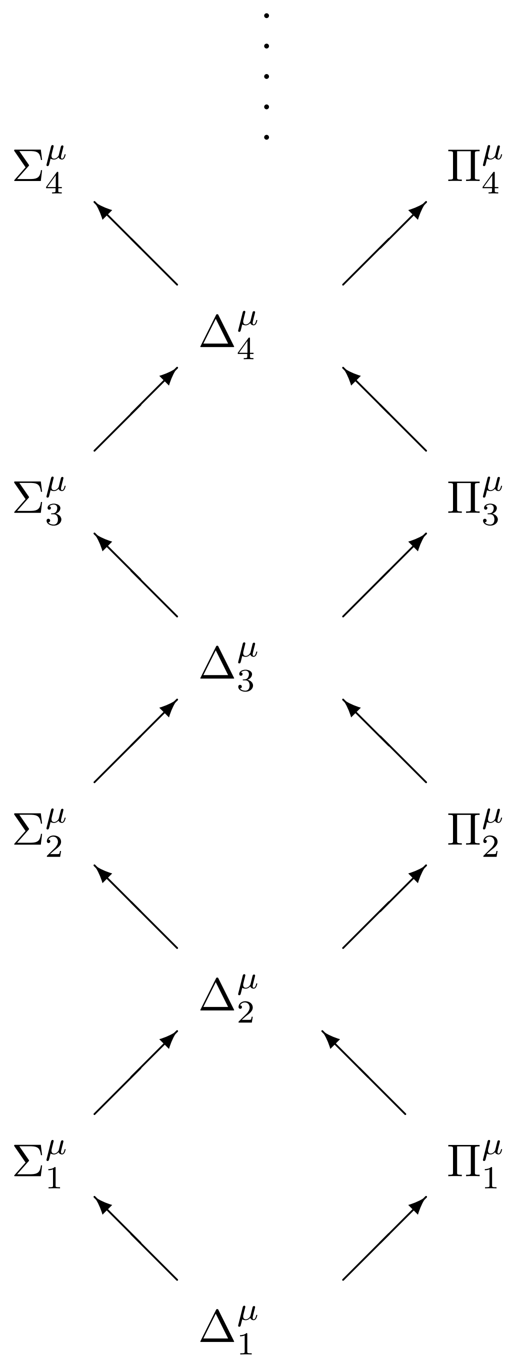
The syntactical fixpoint alternation hierarchy

- $\Sigma_0^\mu = \Pi_0^\mu$ - formulas without fixpoints;
- $\Sigma_{n+1}^\mu = \mu(\Pi_n^\mu)$;
- $\Pi_{n+1}^\mu = \nu(\Sigma_n^\mu)$.

$$\Delta_n^\mu := \Sigma_n^\mu \cap \Pi_n^\mu$$

The syntactical fixpoint alternation hierarchy

Clearly the syntactical fixpoint alternation hierarchy is strict.



Σ_4^μ
Π_4^μ
Δ_4^μ
Σ_3^μ
Π_3^μ
Δ_3^μ
Σ_2^μ
Π_2^μ
Δ_2^μ
Σ_1^μ
Π_1^μ
Δ_1^μ

The syntactical fixpoint alternation hierarchy

$$\mathcal{L}_\mu = \bigcup_{n \in \omega} \Sigma_n^\mu = \bigcup_{n \in \omega} \Pi_n^\mu$$

$$fad(\varphi) = n \text{ iff } n = inf\{k : \varphi \in \Delta_{k+1}^\mu\}$$

The semantical fixpoint alternation hierarchy

$$\Sigma_n^{\mathbf{K}} = \{||\varphi|| : \varphi \in \Sigma_n^{\mu}\}$$

$$\Pi_n^{\mathbf{K}} = \{||\varphi|| : \varphi \in \Pi_n^{\mu}\}$$

$$\Delta_n^{\mathbf{K}} := \Sigma_n^{\mathbf{K}} \cap \Pi_n^{\mathbf{K}}$$

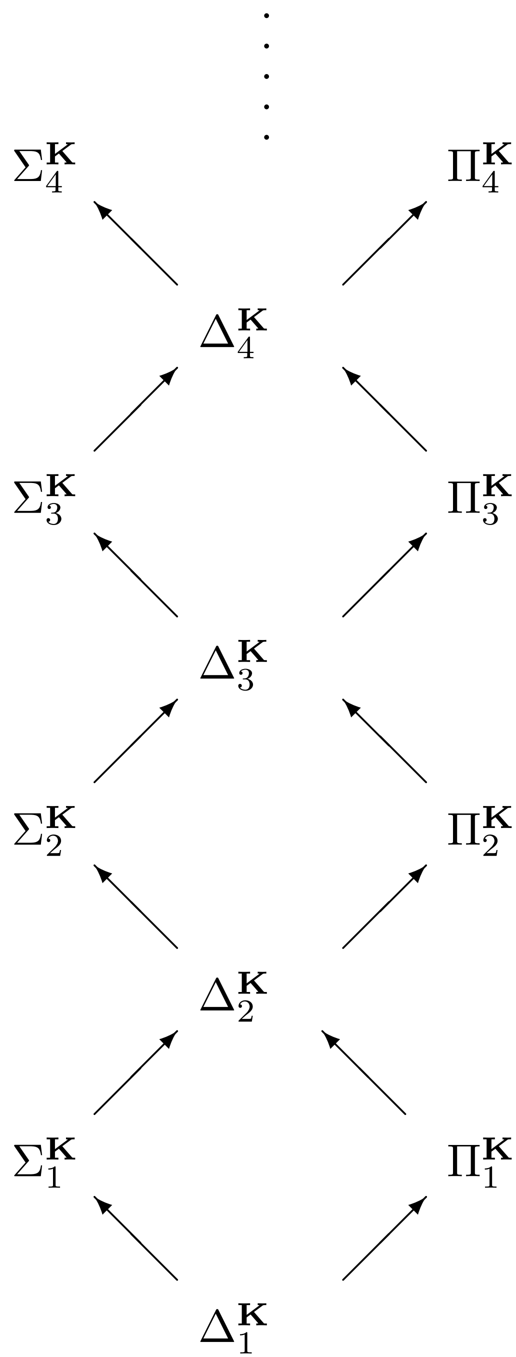
The semantical fixpoint alternation hierarchy

Theorem 2 (Bradfield (96), Lenzi (97)) *The semantical fixpoint alternation depth hierarchy is strict.*

Proof: For every n, the Walukiewicz formula

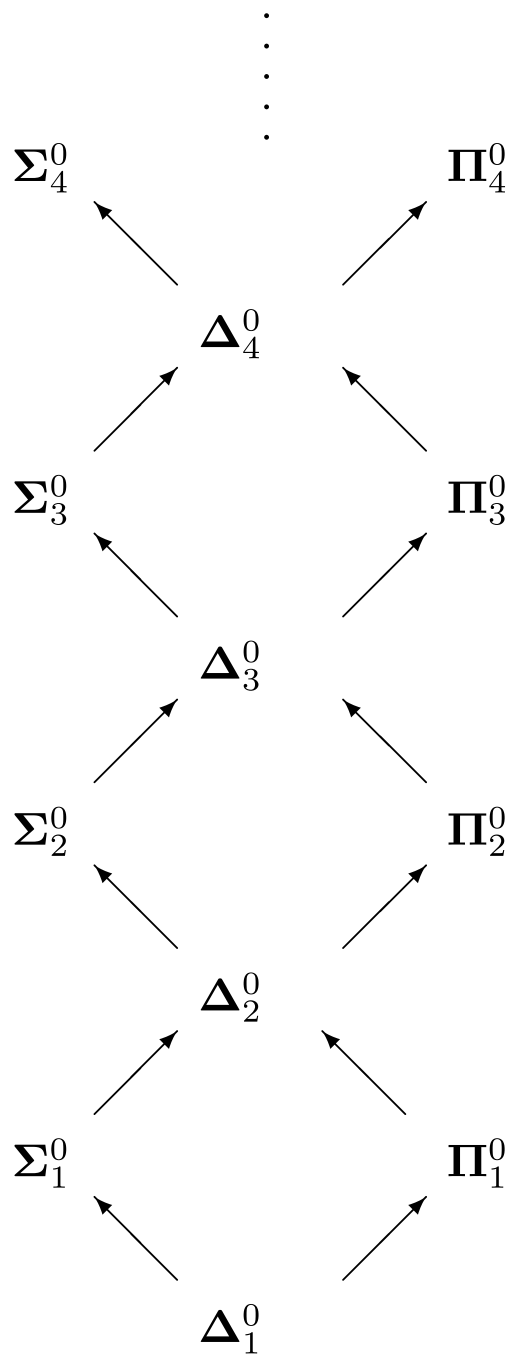
$$\mu x_n \nu x_{n-1} \ldots \mu x_1 \big((p \to \Diamond \bigwedge_{1 \leq n} (r \to x_i)) \wedge (\neg p \to \Box \bigwedge_{1 \leq n} (r \to x_i))\big)$$

is $\Sigma_n^{\mathbf{K}}$-complete.



$\Sigma_4^{\mathbf{K}}$
$\Pi_4^{\mathbf{K}}$
$\Delta_4^{\mathbf{K}}$
$\Sigma_3^{\mathbf{K}}$
$\Pi_3^{\mathbf{K}}$
$\Delta_3^{\mathbf{K}}$
$\Sigma_2^{\mathbf{K}}$
$\Pi_2^{\mathbf{K}}$
$\Delta_2^{\mathbf{K}}$
$\Sigma_1^{\mathbf{K}}$
$\Pi_1^{\mathbf{K}}$
$\Delta_1^{\mathbf{K}}$

. . . but it looks like . . .


$\mathbf{\Sigma}^0_4$
$\mathbf{\Pi}^0_4$
$\mathbf{\Delta}^0_4$
$\mathbf{\Sigma}^0_3$
$\mathbf{\Pi}^0_3$
$\mathbf{\Delta}^0_3$
$\mathbf{\Sigma}^0_2$
$\mathbf{\Pi}^0_2$
$\mathbf{\Delta}^0_2$
$\mathbf{\Sigma}^0_1$
$\mathbf{\Pi}^0_1$
$\mathbf{\Delta}^0_1$

The Borel Hierarchy

As soon as they were introduced, Baire set up the Borel sets in a hierarchy : the Borel hierarchy, which relies on counting how many operations *complementation* and *countable union* are necessary to create a Borel set.

Definition 1

1. $\mathbf{\Sigma}_1^0 = \{open\ sets\}$

2. $\mathbf{\Pi}_\alpha^0 = \{B^\complement : \quad B \in \mathbf{\Sigma}_\alpha^0\}$

3. $\mathbf{\Delta}_\alpha^0 = \mathbf{\Sigma}_\alpha^0 \cap \mathbf{\Pi}_\alpha^0$

4. $\mathbf{\Sigma}_\alpha^0 = \{A = \bigcup_{n \in \mathbb{N}} A_n : \quad A_n \in \bigcup_{\beta < \alpha} \mathbf{\Pi}_\beta^0\}$

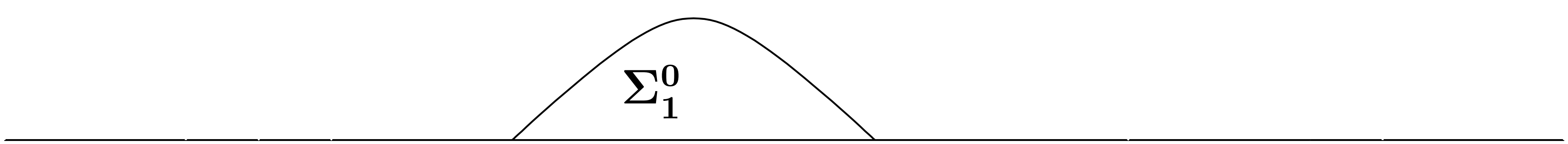
$$\mathcal{B} = \bigcup_{\alpha \in On} \mathbf{\Sigma}_\alpha^0 = \bigcup_{\alpha \in On} \mathbf{\Pi}_\alpha^0 = \bigcup_{\alpha < \omega_1} \mathbf{\Sigma}_\alpha^0 = \bigcup_{\alpha < \omega_1} \mathbf{\Pi}_\alpha^0 = \bigcup_{\alpha < \omega_1} \mathbf{\Delta}_\alpha^0$$

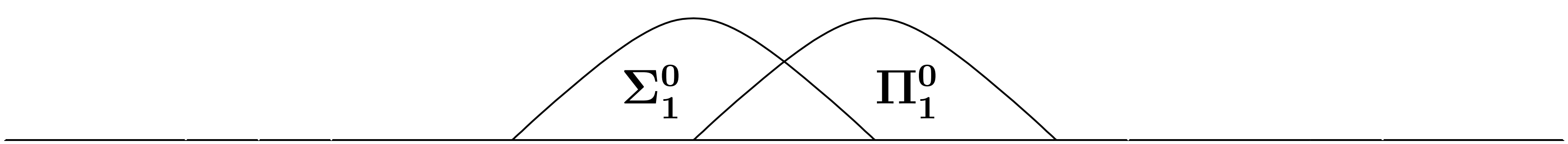
Borel Sets from Above

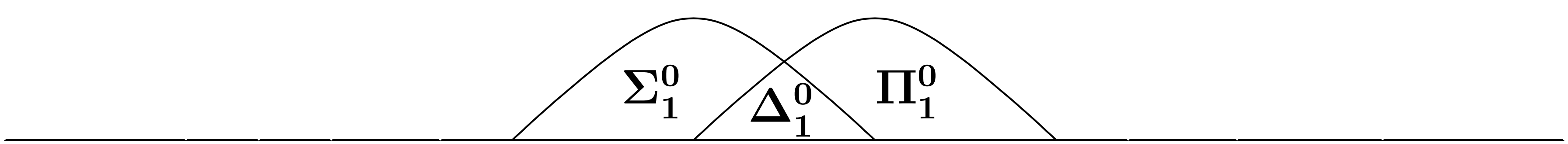
Suslin showed that, for countable alphabets:

$$\mathcal{B} = \bigcup_{\alpha<\omega_1} \boldsymbol{\Sigma}^0_\alpha \;=\; \boldsymbol{\Sigma}^1_1 \cap \boldsymbol{\Pi}^1_1 = \boldsymbol{\Delta}^1_1$$

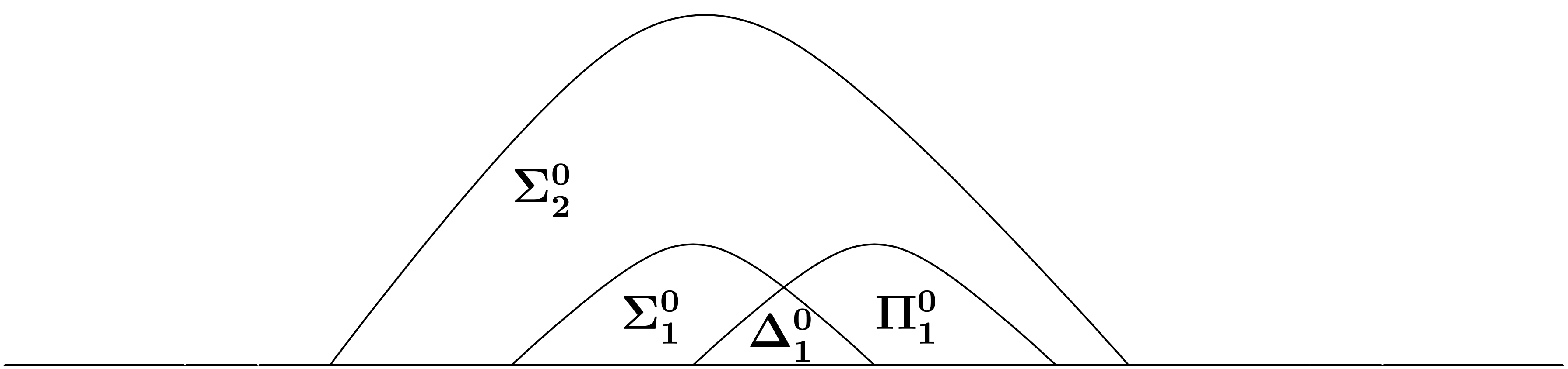
- $\boldsymbol{\Sigma}^1 \;\leftrightsquigarrow\; \exists^{\mathcal{P}(\mathbb{N})}$
- $\boldsymbol{\Pi}^1 = \leftrightsquigarrow\; \forall^{\mathcal{P}(\mathbb{N})}$
- $\mathcal{P}(\mathbb{N}) \;\leftrightsquigarrow\; \mathbb{R} \;\leftrightsquigarrow\; A^\omega$
- $\boldsymbol{\Sigma}^1_1 \;=\; \exists^{\mathcal{P}(\mathbb{N})}\boldsymbol{\Pi}^0_1 \;=\; \exists^{\mathcal{P}(\mathbb{N})}\mathcal{B} \;=\; \exists^{A^\omega}\mathcal{B}$
- $\boldsymbol{\Pi}^1_1 \;=\; \forall^{\mathcal{P}(\mathbb{N})}\mathcal{B} \;=\; \forall^{A^\omega}\mathcal{B}$

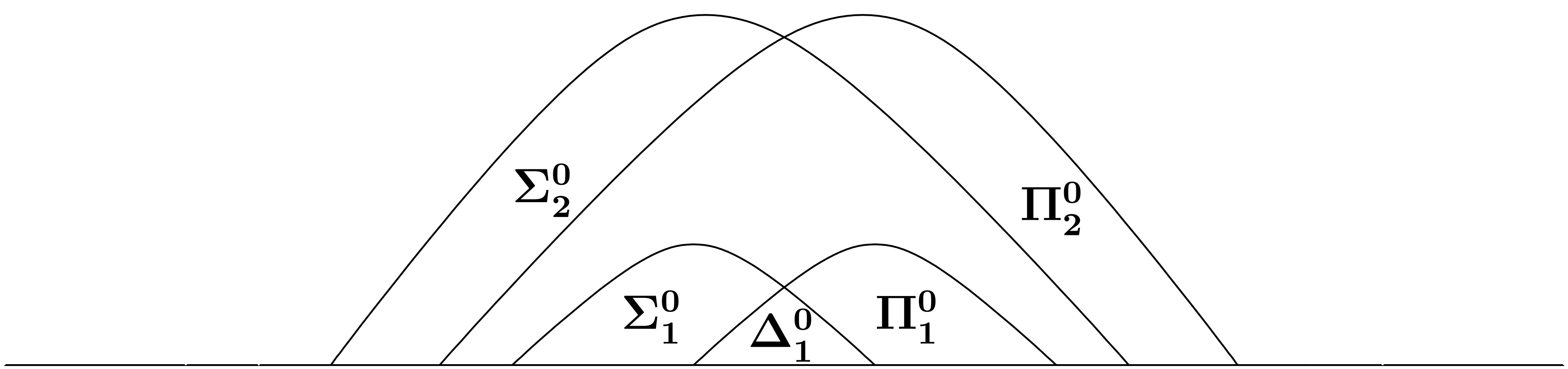

Σ_1^0

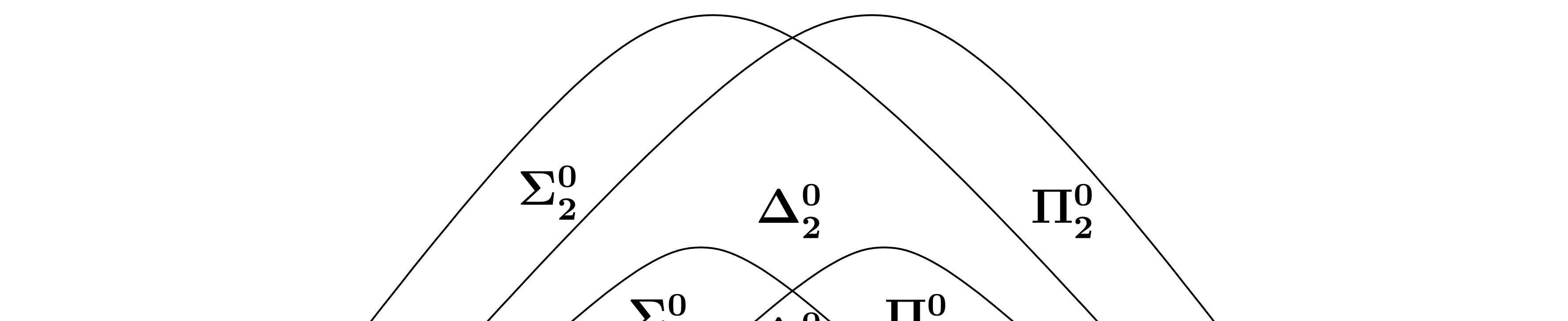

Σ_1^0
Π_1^0



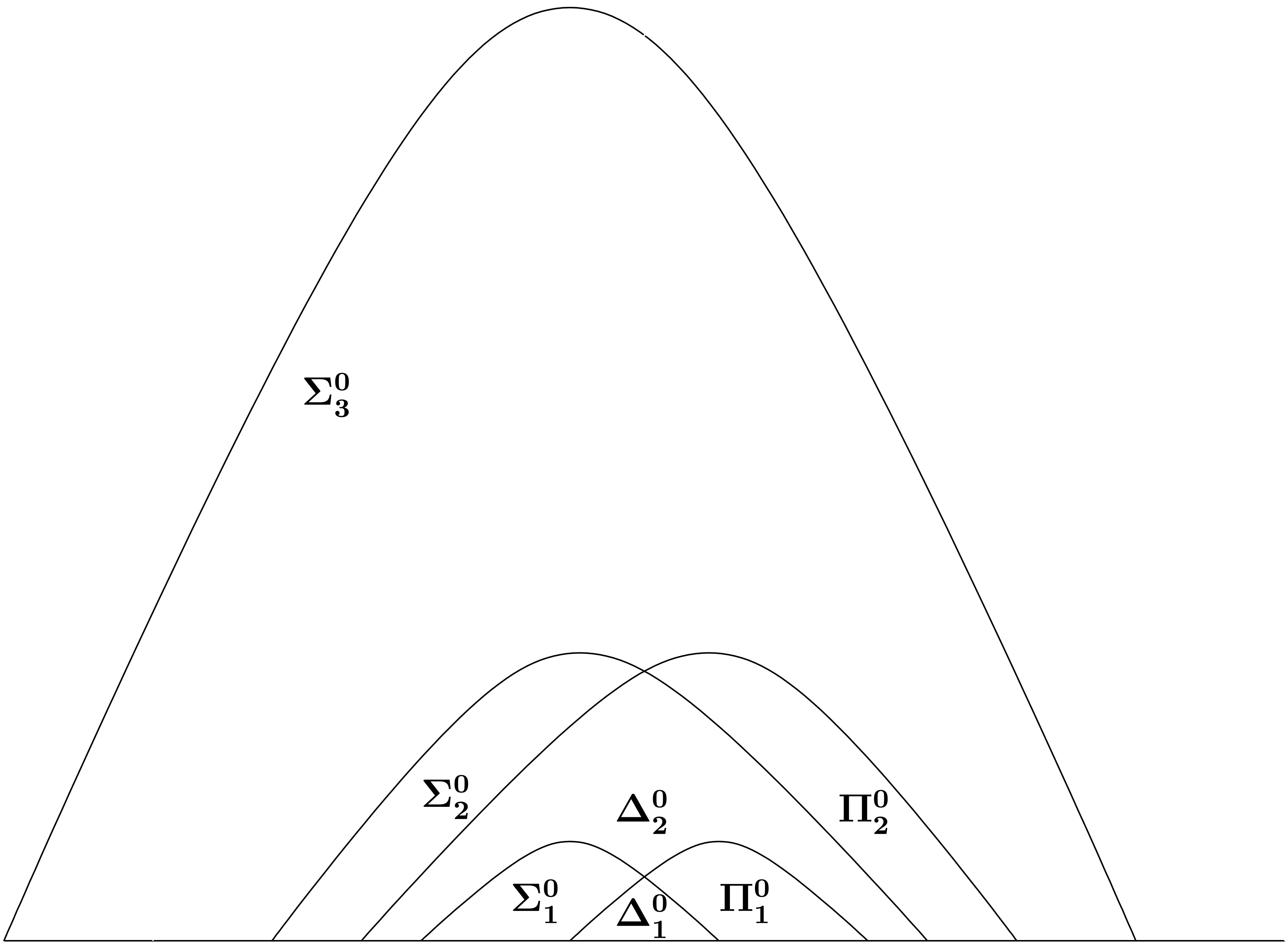
Σ_1^0
Δ_1^0
Π_1^0


$\boldsymbol{\Sigma}^0_2$
$\boldsymbol{\Sigma}^0_1$
$\boldsymbol{\Delta}^0_1$
$\boldsymbol{\Pi}^0_1$

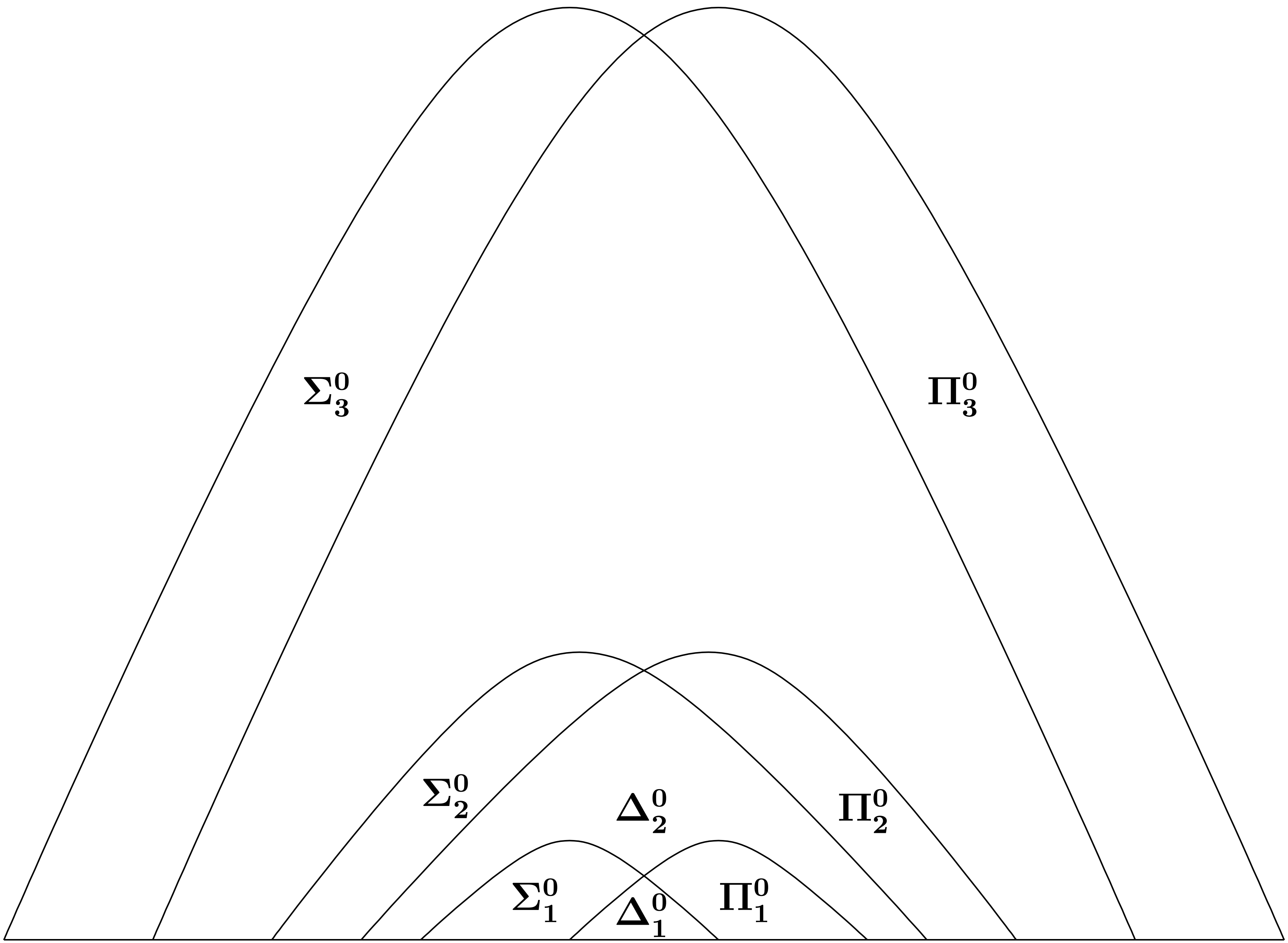

$\mathbf{\Sigma}_2^0$
$\mathbf{\Pi}_2^0$
$\mathbf{\Sigma}_1^0$
$\mathbf{\Delta}_1^0$
$\mathbf{\Pi}_1^0$



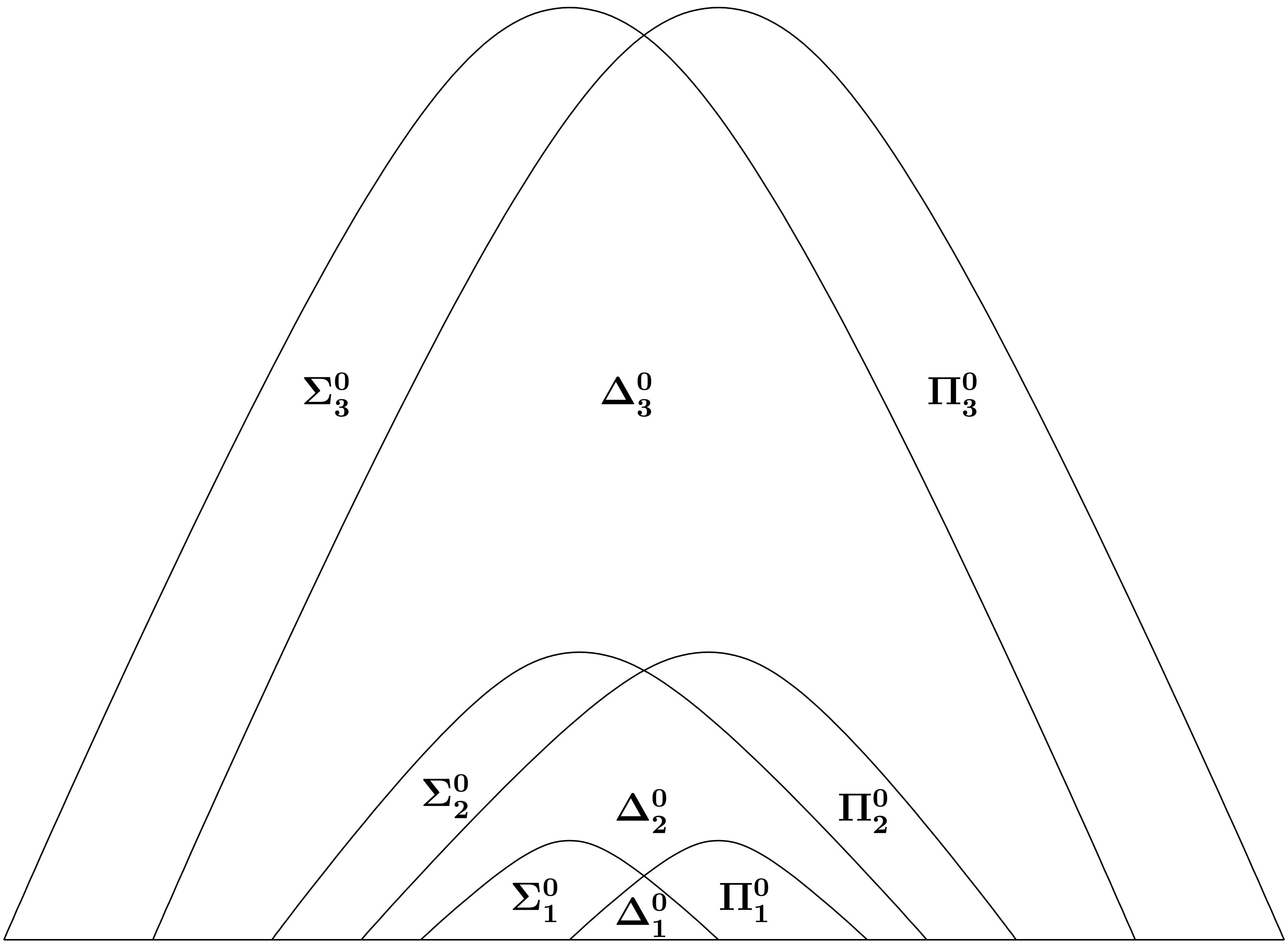
$\mathbf{\Sigma}^0_2$
$\mathbf{\Delta}^0_2$
$\mathbf{\Pi}^0_2$
$\mathbf{\Sigma}^0_1$
$\mathbf{\Delta}^0_1$
$\mathbf{\Pi}^0_1$



$\boldsymbol{\Sigma}^0_3$
$\boldsymbol{\Sigma}^0_2$
$\boldsymbol{\Delta}^0_2$
$\boldsymbol{\Pi}^0_2$
$\boldsymbol{\Sigma}^0_1$
$\boldsymbol{\Delta}^0_1$
$\boldsymbol{\Pi}^0_1$



Σ_3^0
Π_3^0
Σ_2^0
Δ_2^0
Π_2^0
Σ_1^0
Δ_1^0
Π_1^0


$\mathbf{\Sigma}^0_3$
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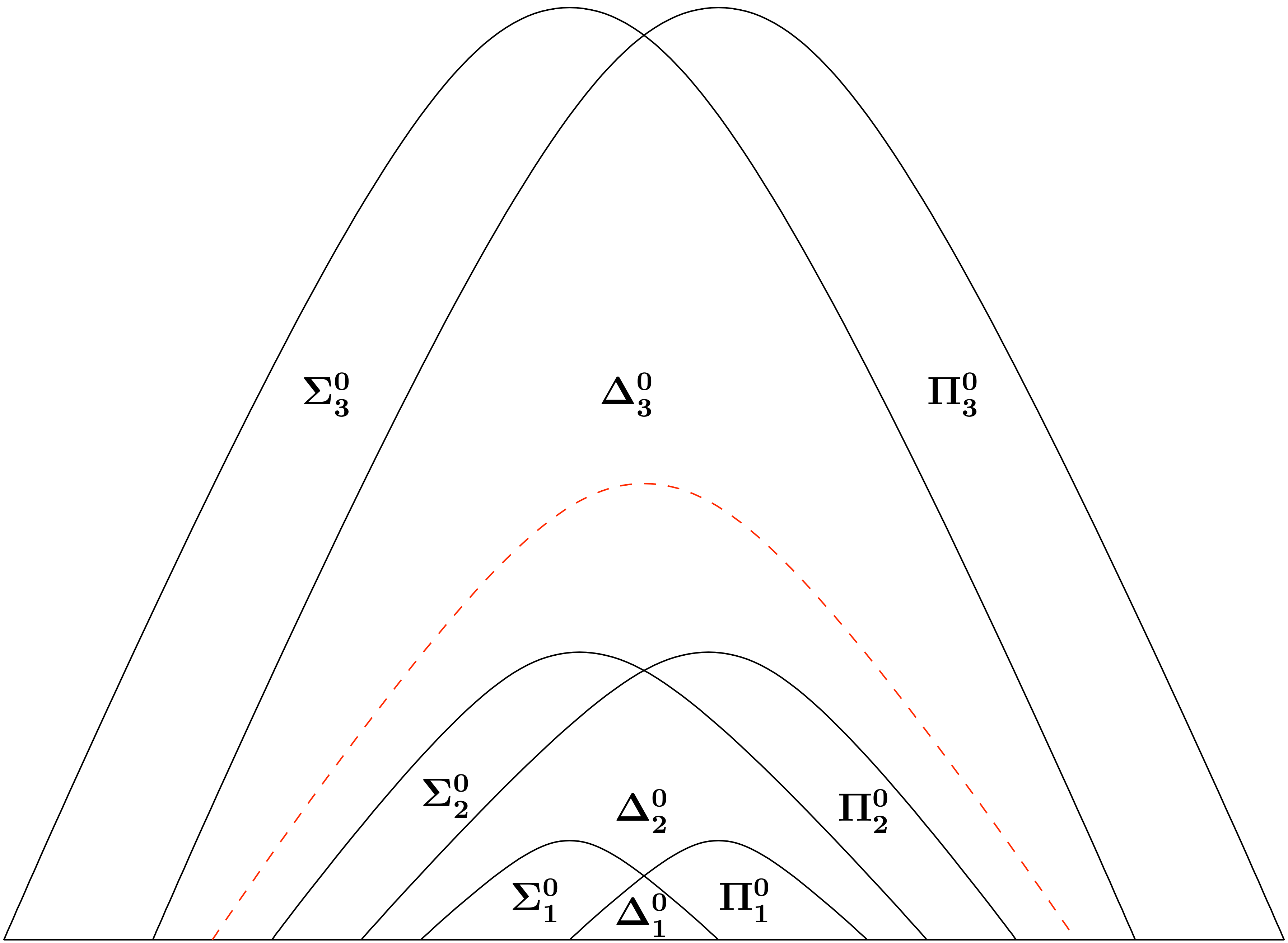
$\mathbf{\Delta}^0_1$
$\mathbf{\Pi}^0_1$



$\mathbf{\Sigma^0_3}$
$\mathbf{\Delta^0_3}$
$\mathbf{\Pi^0_3}$
$\mathbf{\Sigma^0_2}$
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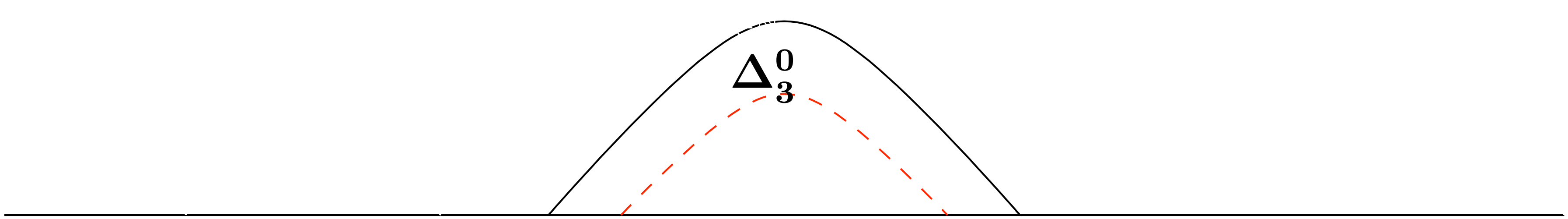
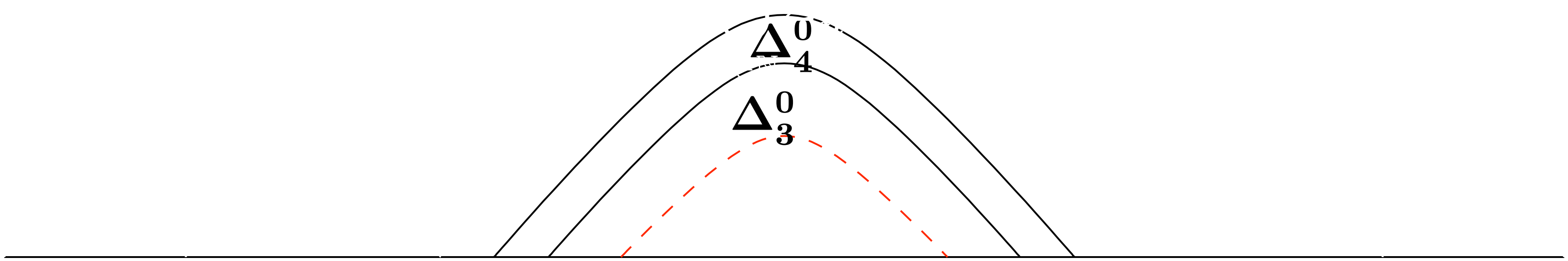
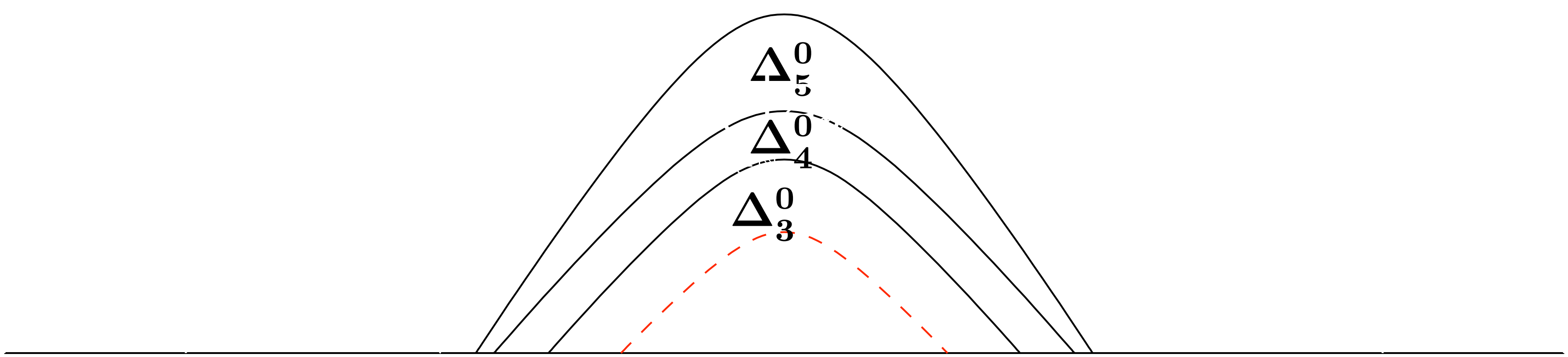
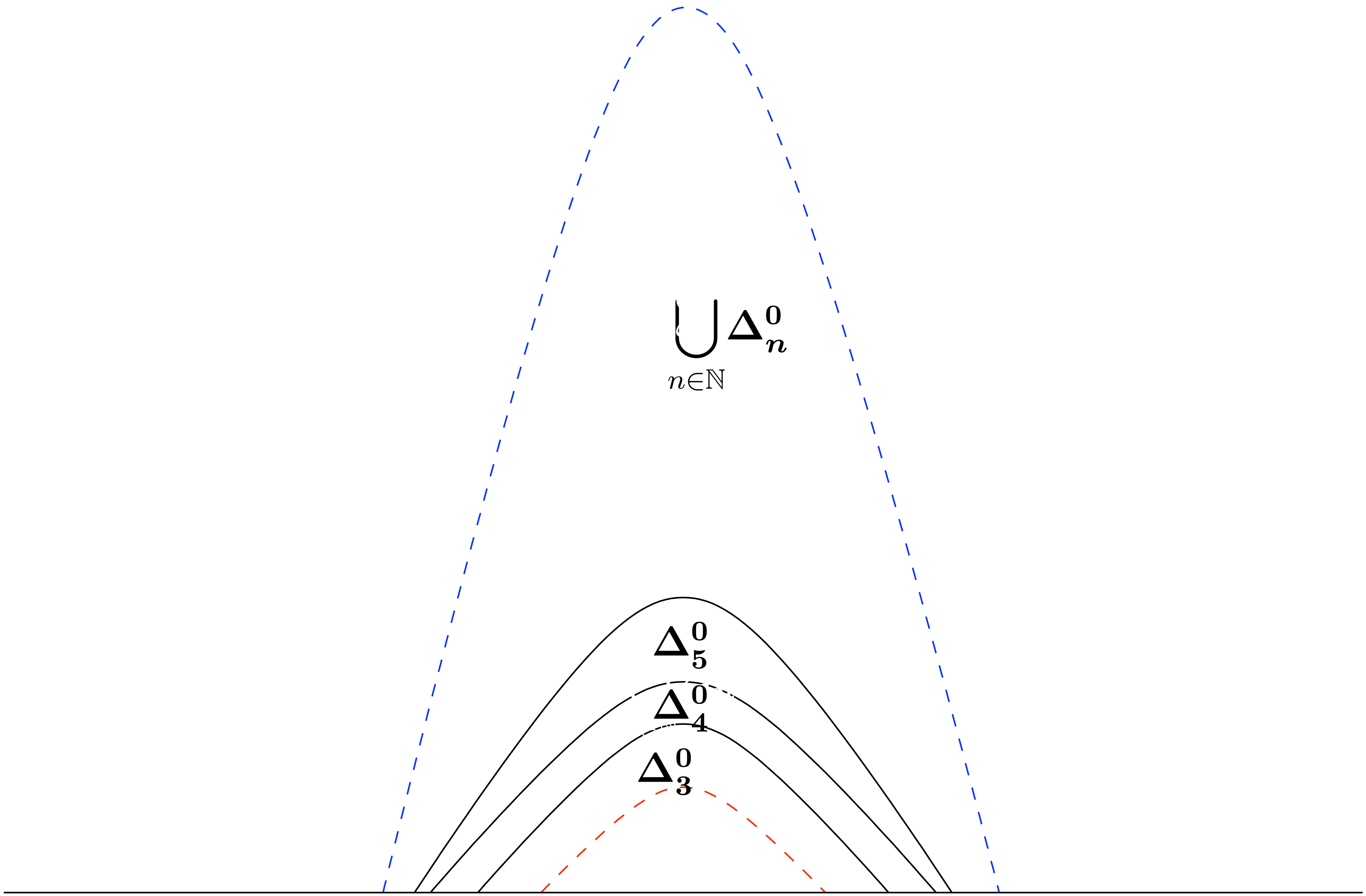
$\mathbf{\Sigma^0_1}$
$\mathbf{\Delta^0_1}$
$\mathbf{\Pi^0_1}$


Δ_3^0

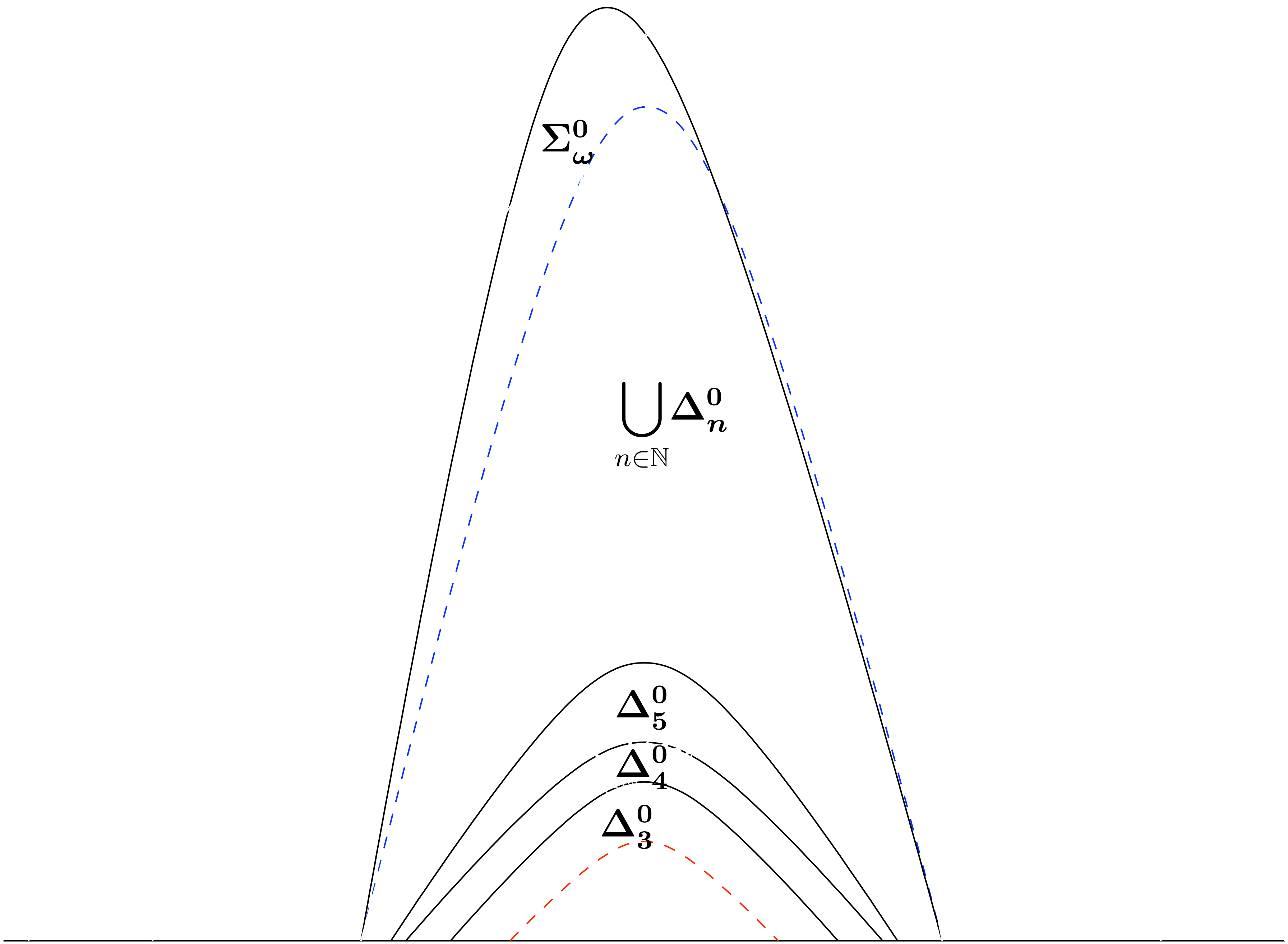


Δ_4^0
Δ_3^0

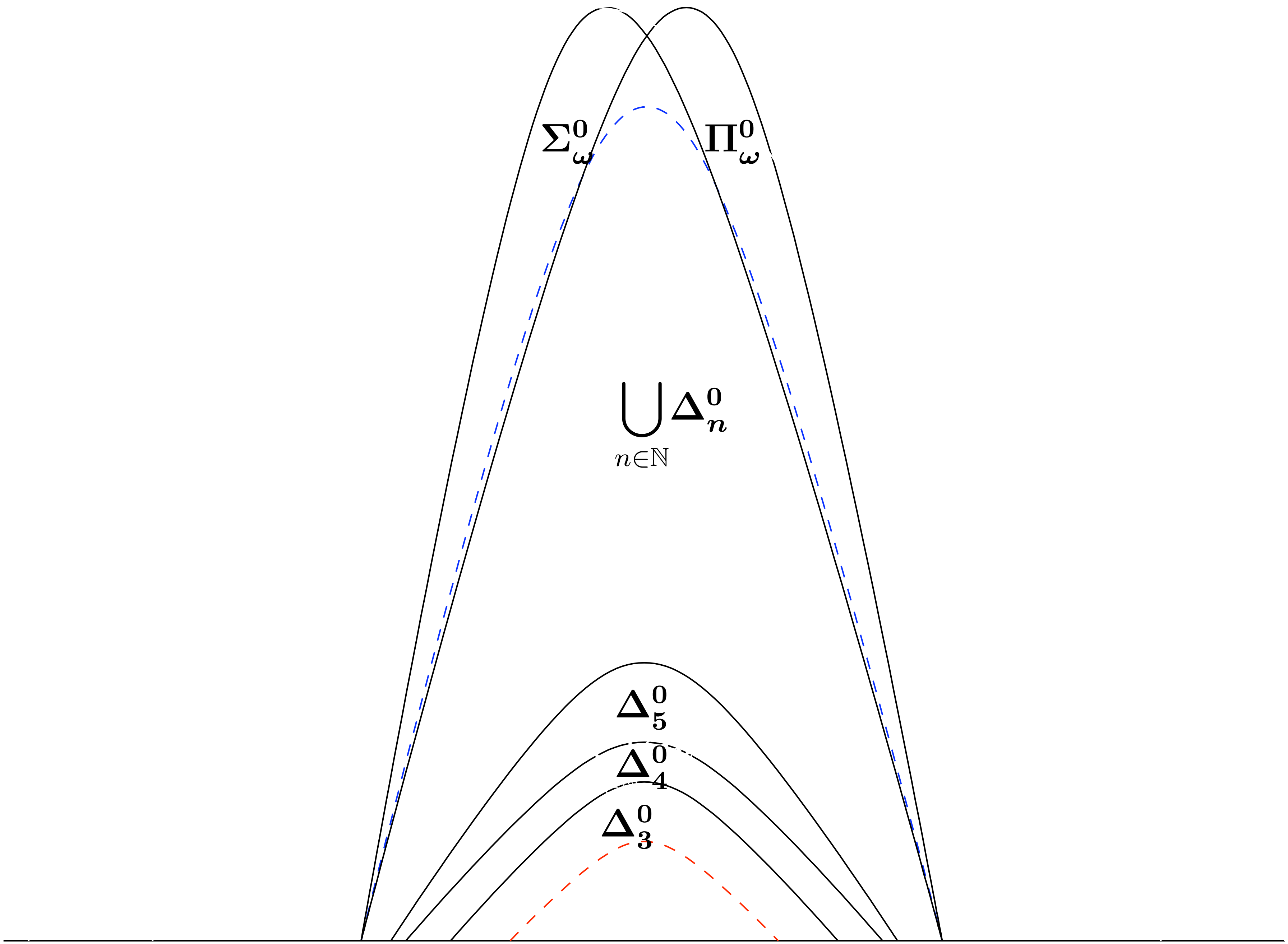

Δ_5^0
Δ_4^0
Δ_3^0



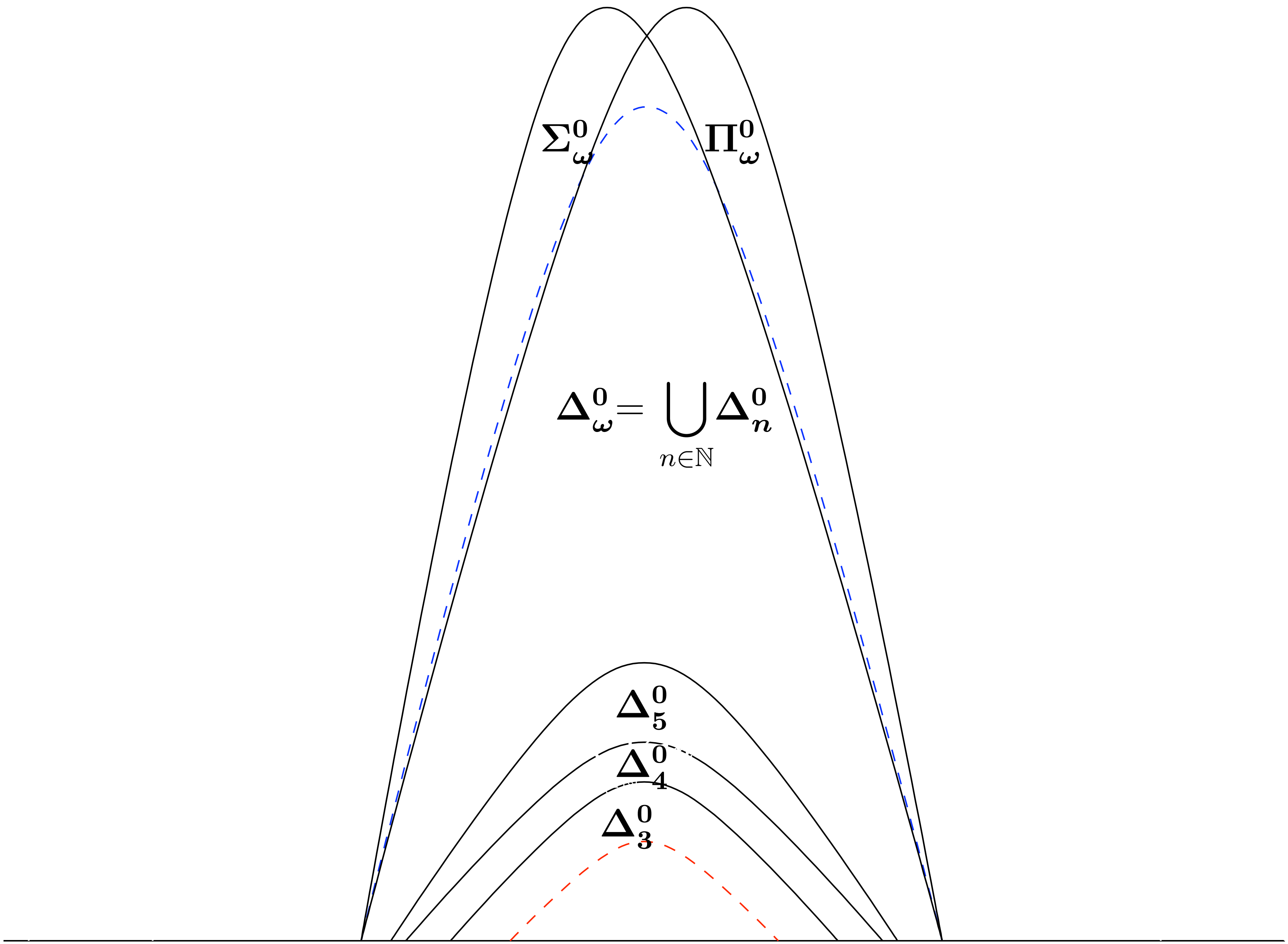
$\bigcup_{n\in\mathbb{N}} \Delta^0_n$
Δ^0_5
Δ^0_4
Δ^0_3

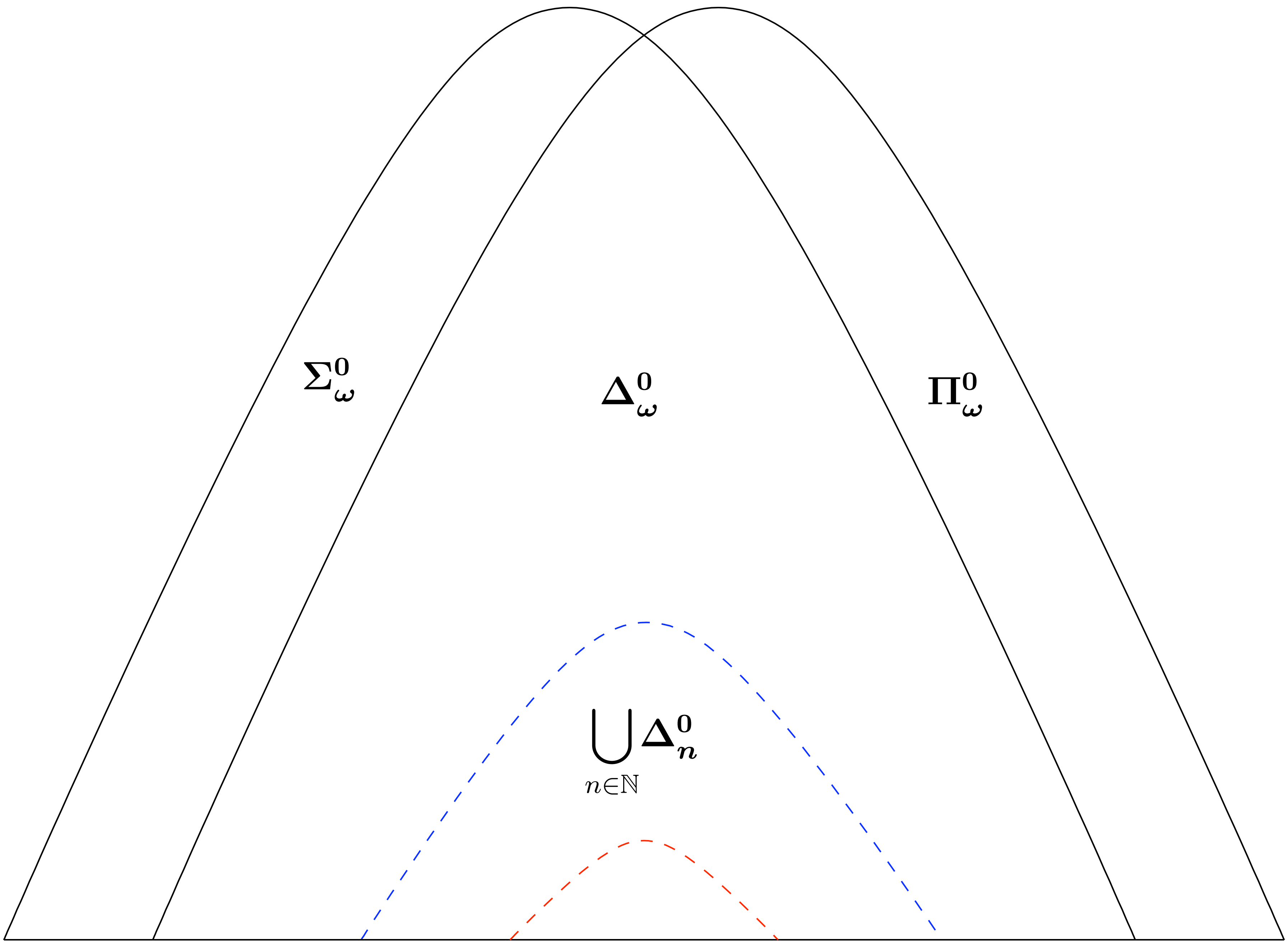


$\mathbf{\Sigma}^0_{\boldsymbol{\omega}}$
$\bigcup_{n\in\mathbb{N}} \mathbf{\Delta}^0_{\boldsymbol{n}}$
$\mathbf{\Delta}^0_5$
$\mathbf{\Delta}^0_4$
$\mathbf{\Delta}^0_3$

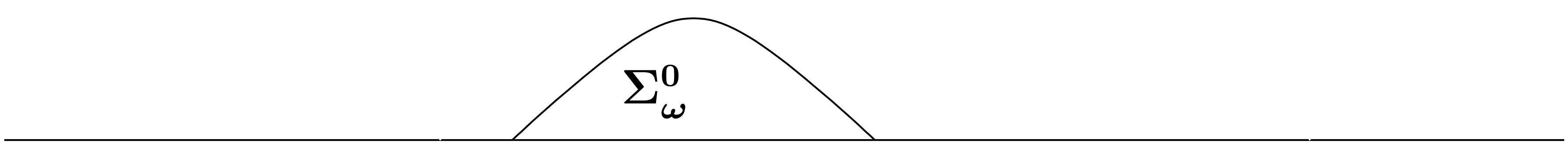


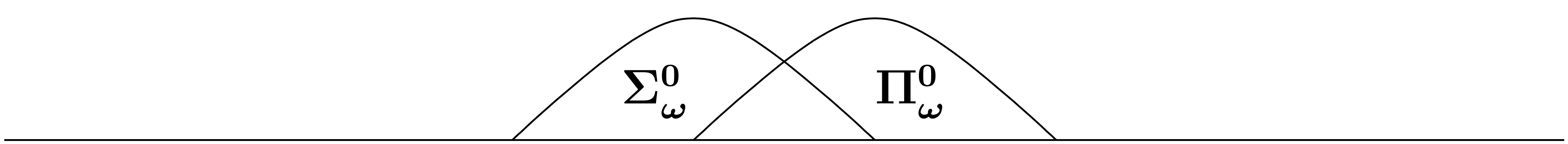
$\mathbf{\Sigma}^0_{\boldsymbol{\omega}}$
$\mathbf{\Pi}^0_{\boldsymbol{\omega}}$
$\bigcup_{n\in\mathbb{N}} \mathbf{\Delta}^0_{\boldsymbol{n}}$
$\mathbf{\Delta}^0_5$
$\mathbf{\Delta}^0_4$
$\mathbf{\Delta}^0_3$

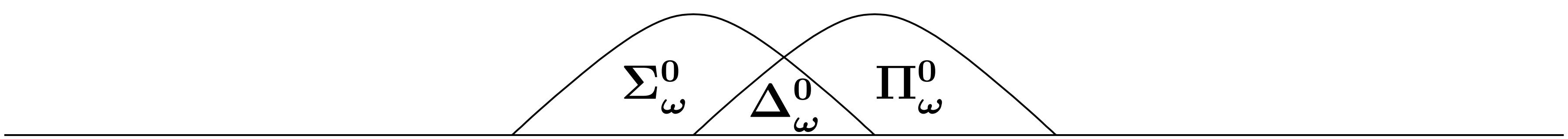

$\mathbf{\Sigma^0_\omega}$
$\mathbf{\Pi^0_\omega}$
$\mathbf{\Delta^0_\omega} = \bigcup_{n\in\mathbb{N}} \mathbf{\Delta^0_n}$
$\mathbf{\Delta^0_5}$
$\mathbf{\Delta^0_4}$
$\mathbf{\Delta^0_3}$

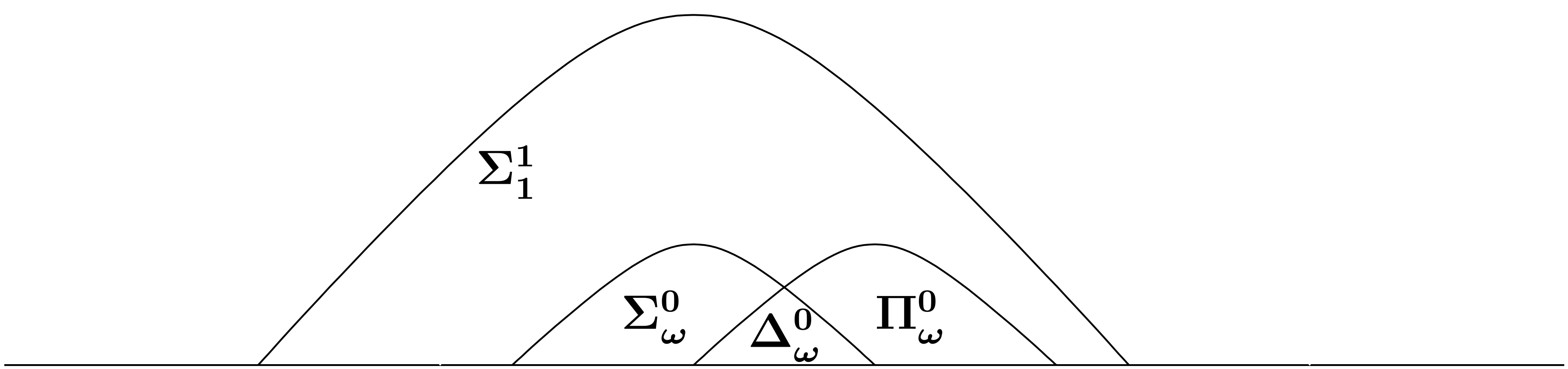


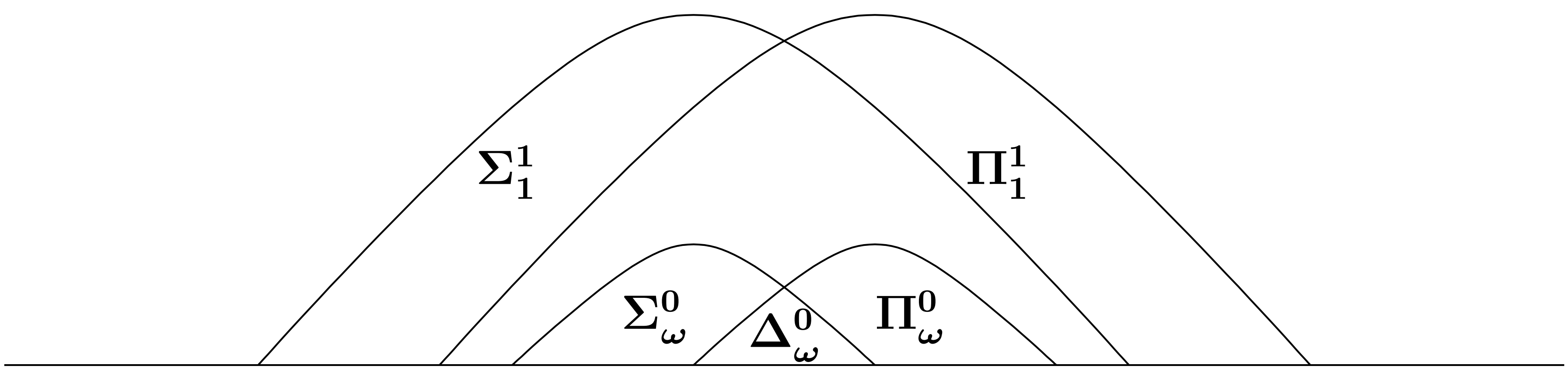
$\mathbf{\Sigma}^0_{\omega}$
$\mathbf{\Delta}^0_{\omega}$
$\mathbf{\Pi}^0_{\omega}$
$\bigcup_{n\in\mathbb{N}} \mathbf{\Delta}^0_{n}$

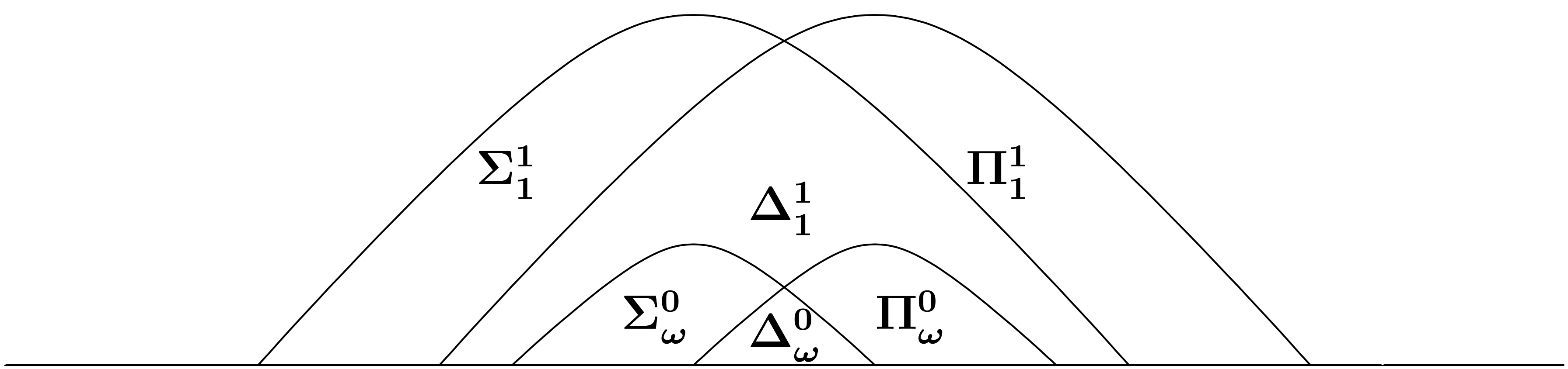

Σ^0_ω


Σ^0_ω
Π^0_ω

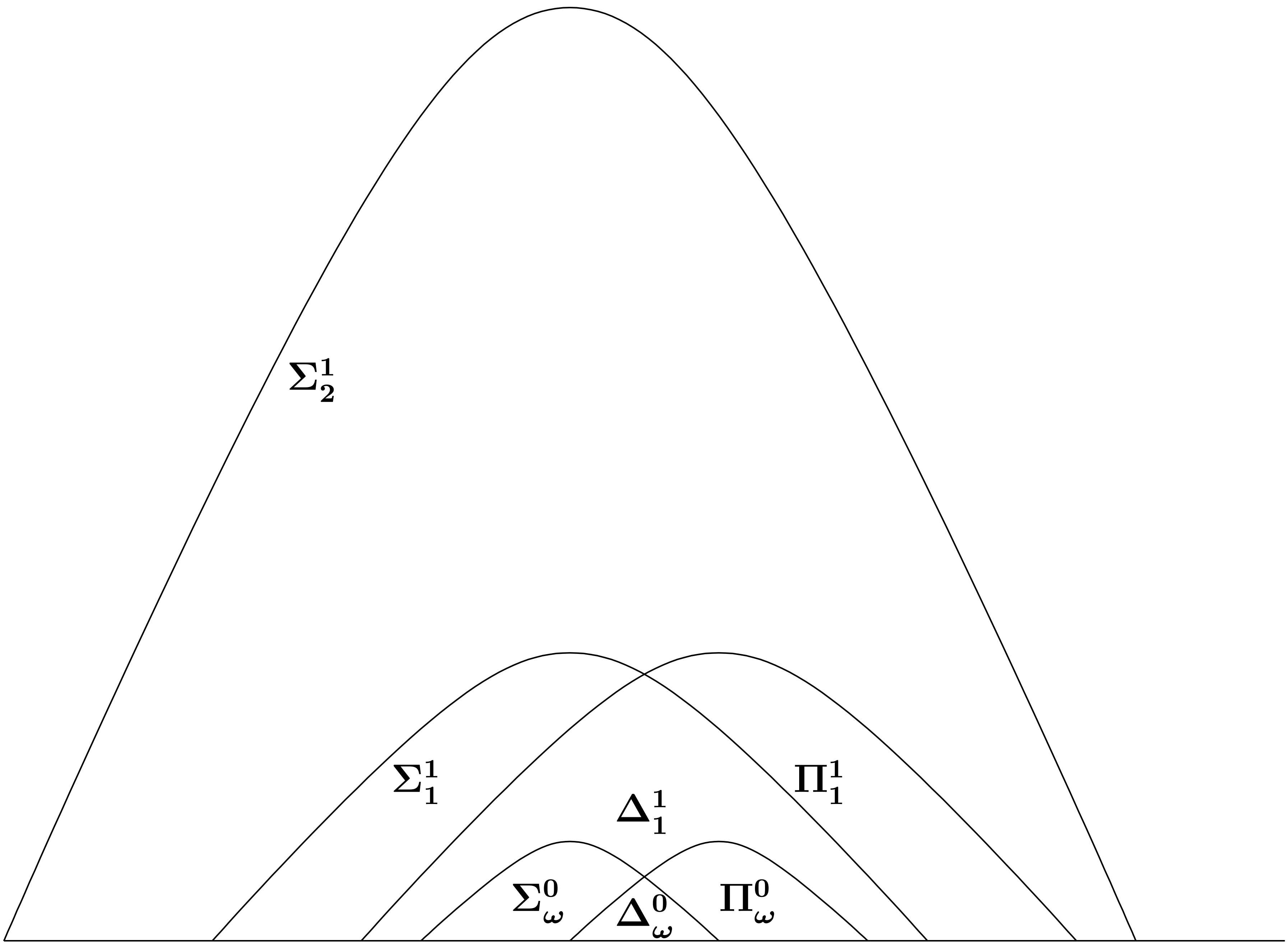

Σ^0_ω
Δ^0_ω
Π^0_ω


Σ^1_1
Σ^0_ω
Δ^0_ω
Π^0_ω

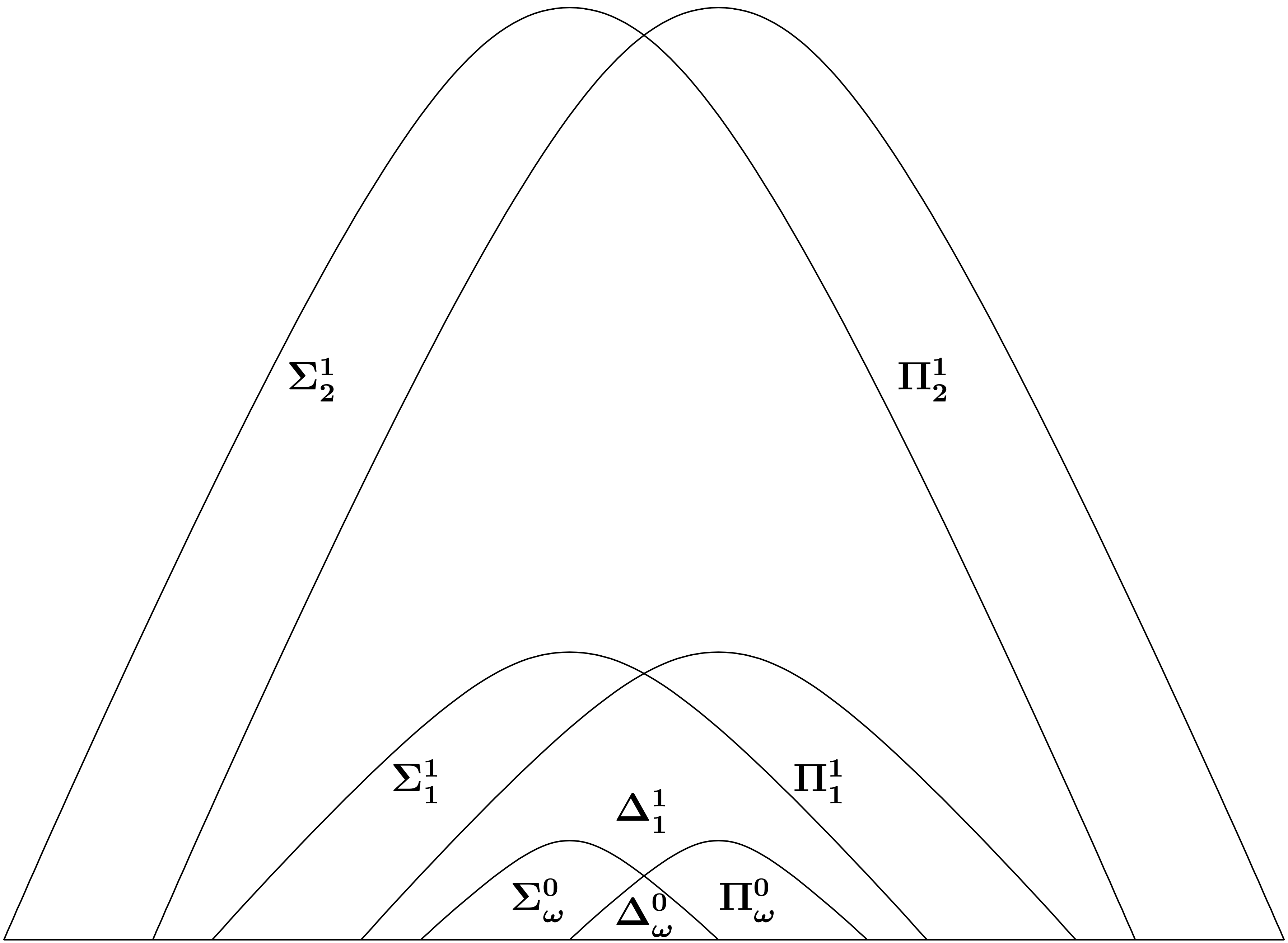

Σ^1_1
Π^1_1
Σ^0_ω
Δ^0_ω
Π^0_ω



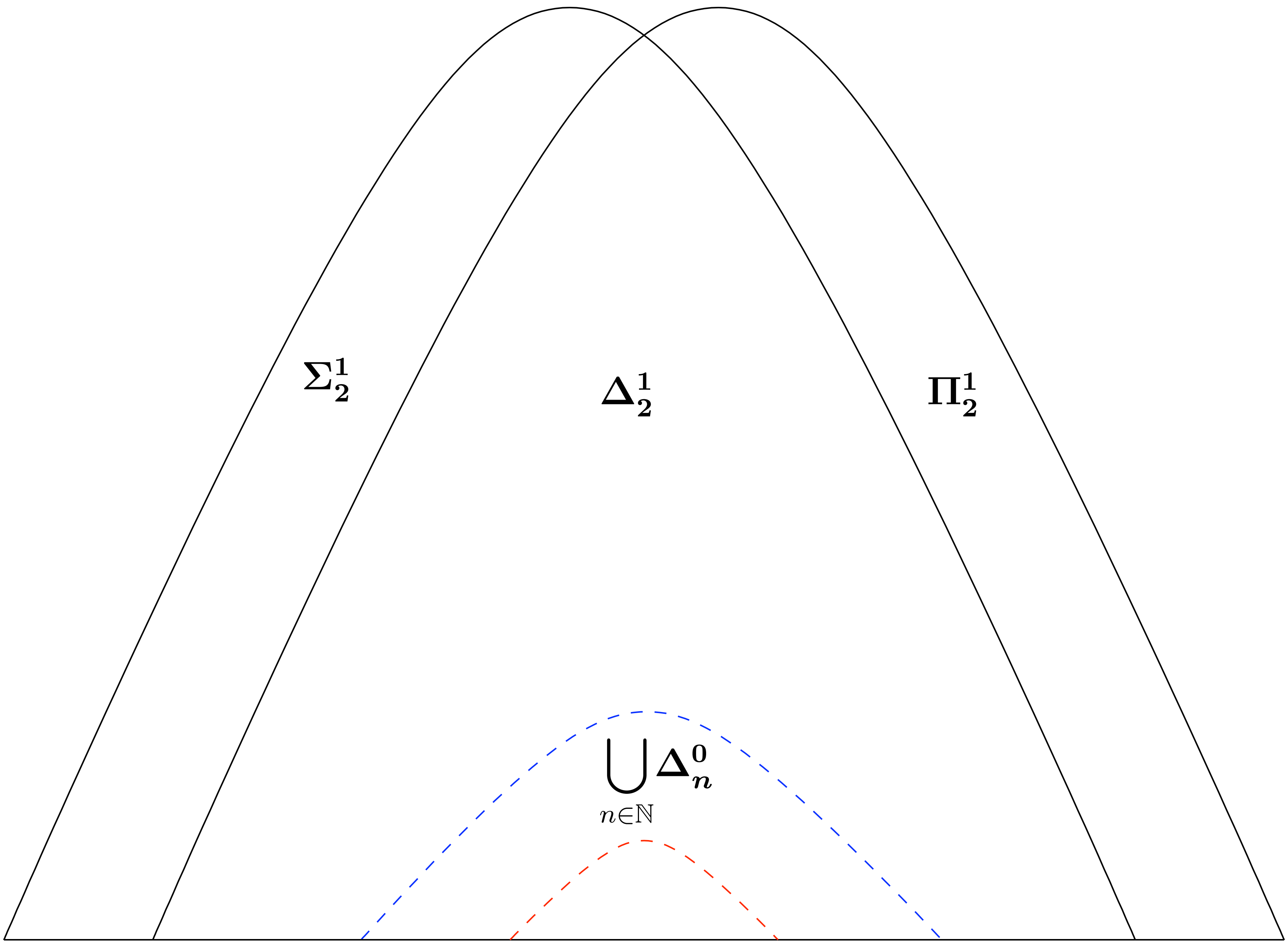
Σ^1_1
Π^1_1
Δ^1_1
Σ^0_ω
Δ^0_ω
Π^0_ω



$\mathbf{\Sigma}^1_2$
$\mathbf{\Sigma}^1_1$
$\mathbf{\Pi}^1_1$
$\mathbf{\Delta}^1_1$
$\mathbf{\Sigma}^0_\omega$
$\mathbf{\Delta}^0_\omega$
$\mathbf{\Pi}^0_\omega$



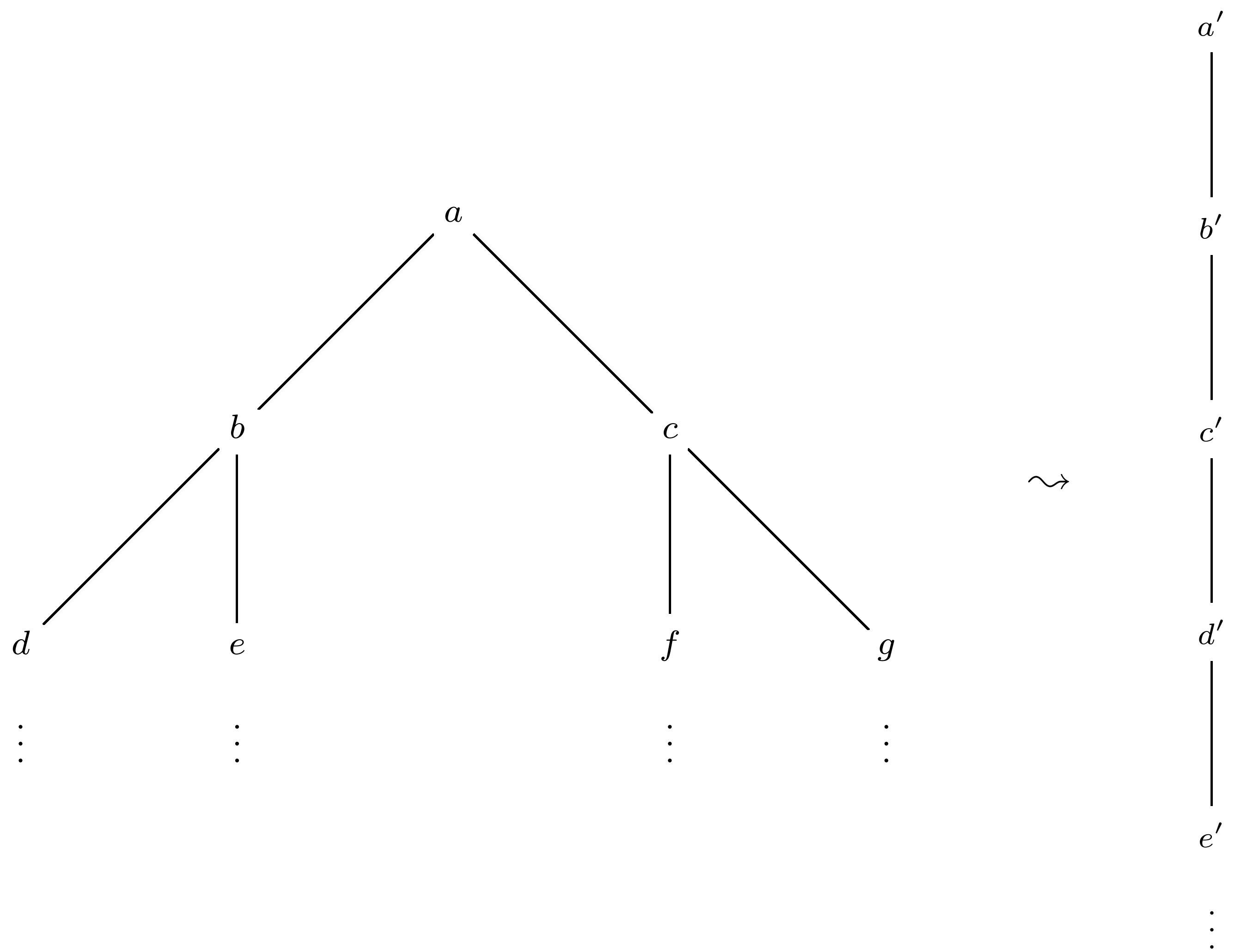
$\mathbf{\Sigma}^1_2$
$\mathbf{\Pi}^1_2$
$\mathbf{\Sigma}^1_1$
$\mathbf{\Pi}^1_1$
$\mathbf{\Delta}^1_1$
$\mathbf{\Sigma}^0_\omega$
$\mathbf{\Delta}^0_\omega$
$\mathbf{\Pi}^0_\omega$


$\boldsymbol{\Sigma}^1_2$
$\boldsymbol{\Delta}^1_2$
$\boldsymbol{\Pi}^1_2$
$\bigcup_{n\in\mathbb{N}} \boldsymbol{\Delta}^0_n$

What about a hierarchy for Δ^1_2 sets?

We think μ-calculus can help…

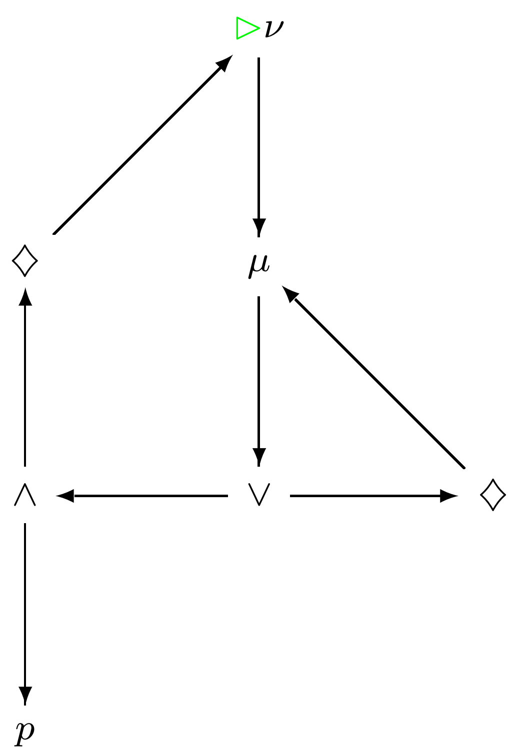
Sets of models of μ-formulae are sets of labelled infinite trees. Hence, they can be seen as sets of reals.



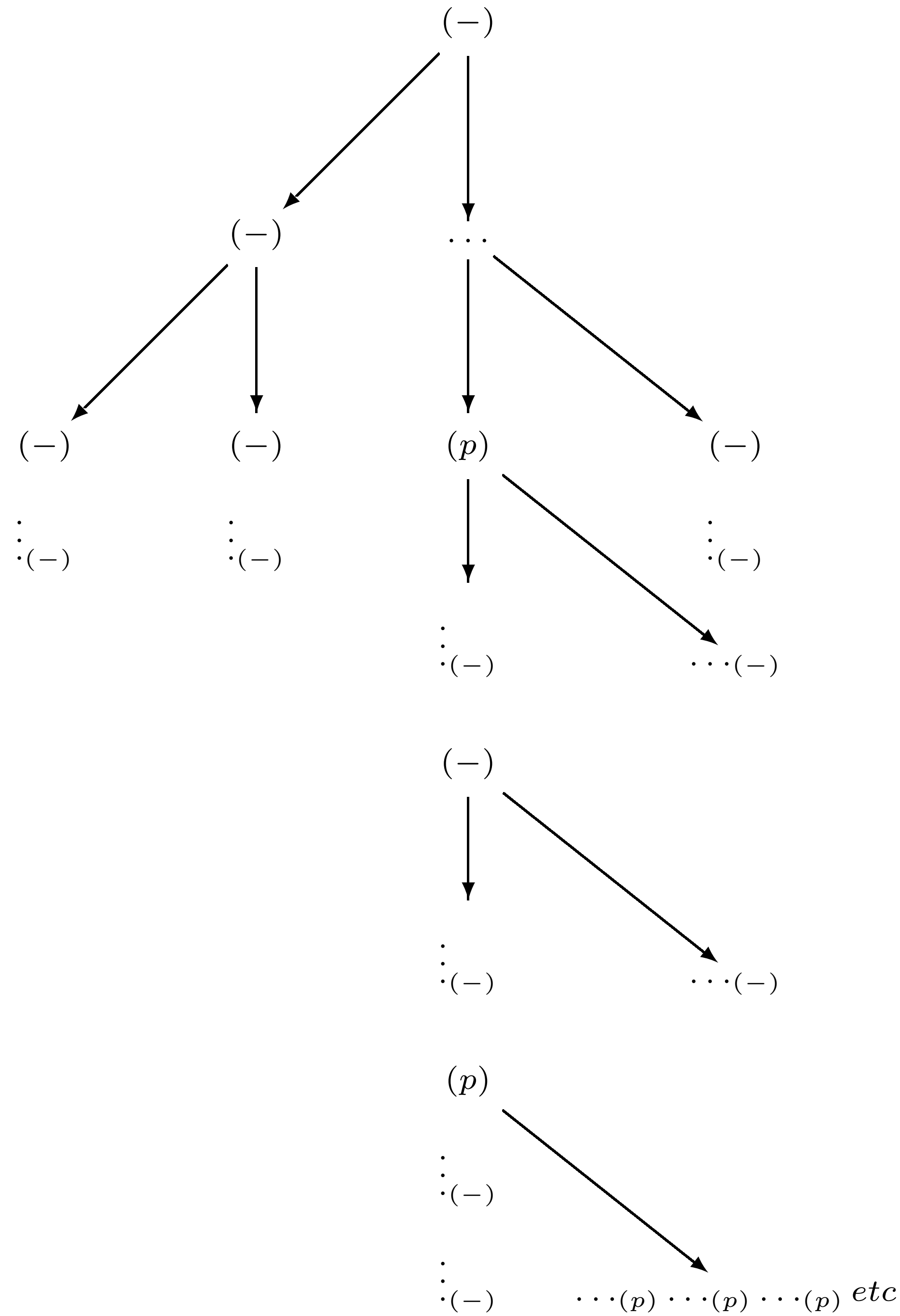
- sets of trees regarded as Cantor or Baire space

Example: Σ_1^1-complete

$$\nu x\big(\mu y((p \wedge \Diamond x) \vee \Diamond y)\big)$$



▷ν
μ
◊
∧
∨
◊
p



(−)
(−)
(−)
(−)
(p)
(−)
(p)
etc

For every $n \geq 2$, sets of models of Walukiewicz formulae

$$\mu x_n \nu x_{n-1} \ldots \mu x_1 \Big((p \to \Diamond \bigwedge_{1 \leq n} (r \to x_i)) \wedge (\neg p \to \Box \bigwedge_{1 \leq n} (r \to x_i)) \Big)$$

are in $\Delta^1_2 \setminus \Delta^1_1$ ...

For every $n \geq 2$, sets of models of Walukiewicz formulae

$$\mu x_n \nu x_{n-1} \dots \mu x_1 \big((p \rightarrow \Diamond \bigwedge_{1 \leq n} (r \rightarrow x_i)) \wedge (\neg p \rightarrow \Box \bigwedge_{1 \leq n} (r \rightarrow x_i))\big)$$

are in $\Delta^1_2 \setminus \Delta^1_1$ ...

... but where are they located?

- This gives a hint on what a hierarchy of $\Delta^1_2 \setminus \Delta^1_1$ sets might be.

- There exists already a hierarchy of $\Delta^1_2 \setminus \Delta^1_1$ (under determinacy). It has not been described though.

Wadge game $G(L, M)$

Spoiler s t Duplicator

Duplicator may skip his turn, but not forever.

Duplicator wins iff $t \in L \iff s \in M$.

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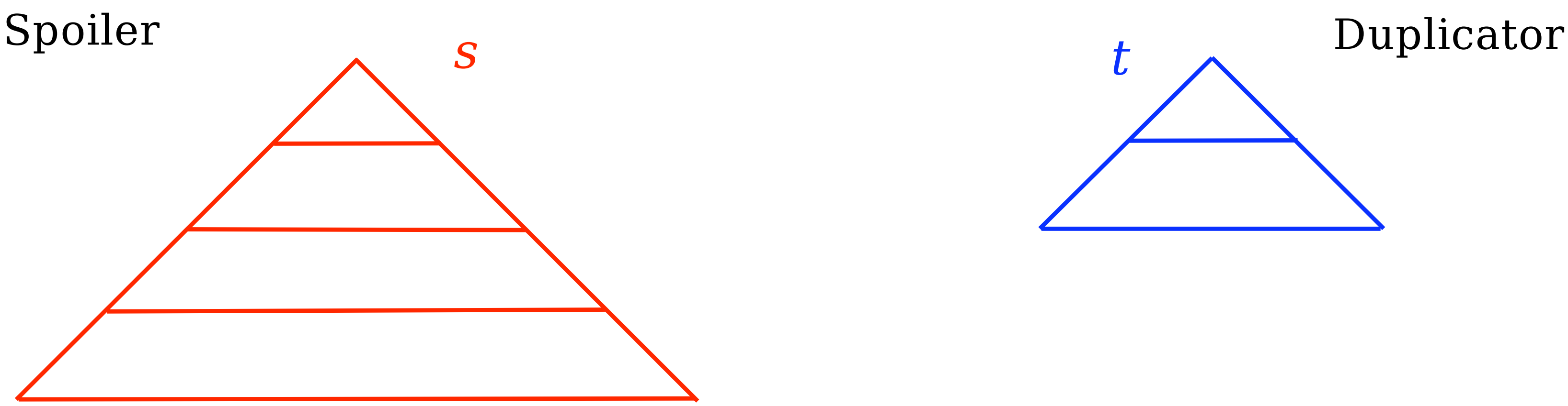

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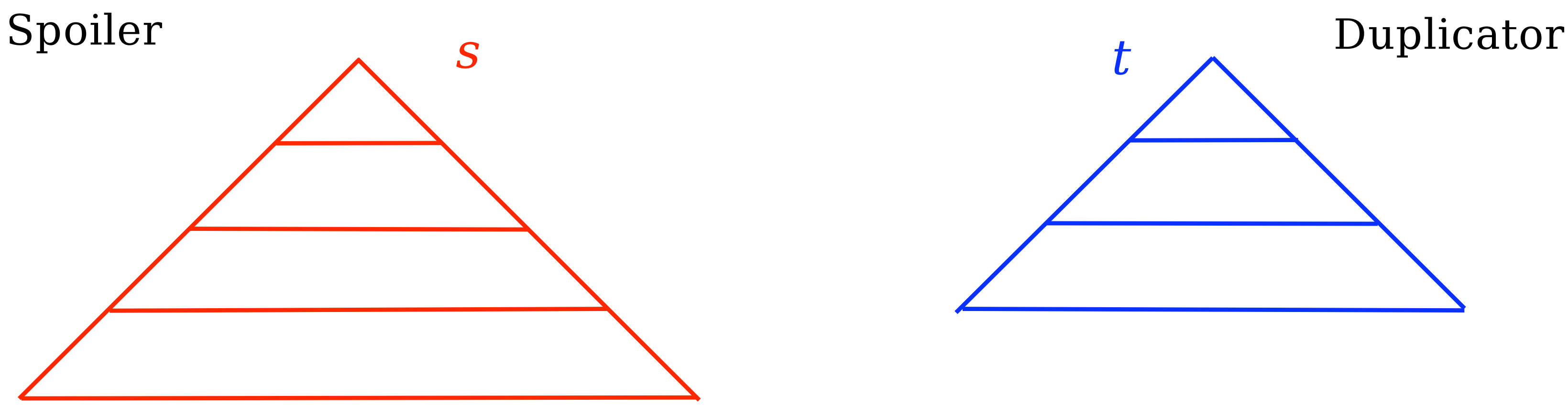

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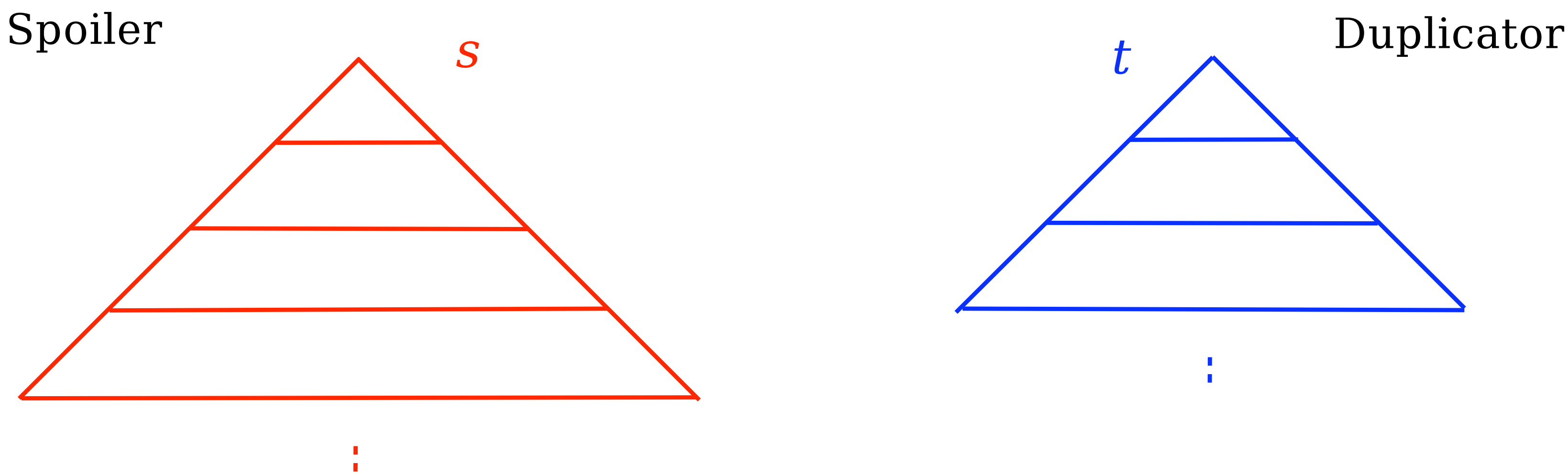

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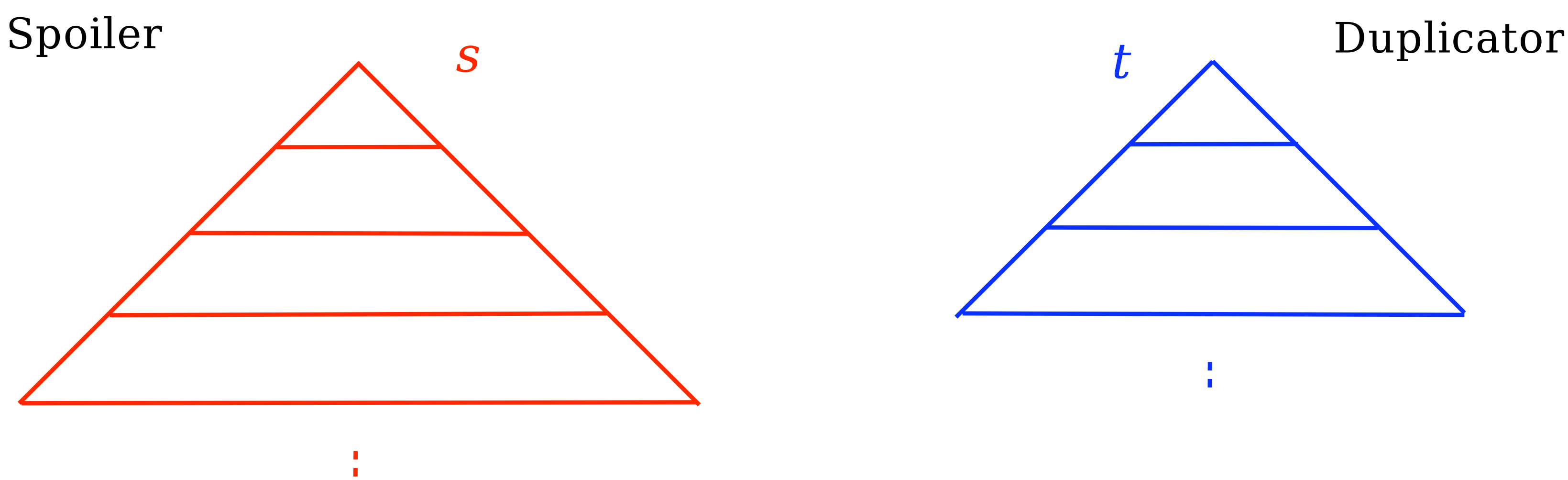
Wadge game $G(L, M)$



Duplicator may skip his turn, but not forever.

Duplicator wins iff $t \in L \iff s \in M$.

$L \leq M$ iff Duplicator has a winning strategy in $G(L, M)$

Wadge game $G(L, M)$

$L \leq M$ iff Duplicator has a winning strategy in $G(L, M)$

$L \leq M$ iff there exists a continuous f s.t. $L = f^{-1}M$

Wadge hierarchy

$L \leq M$ iff Duplicator has a winning strategy in $G(L, M)$

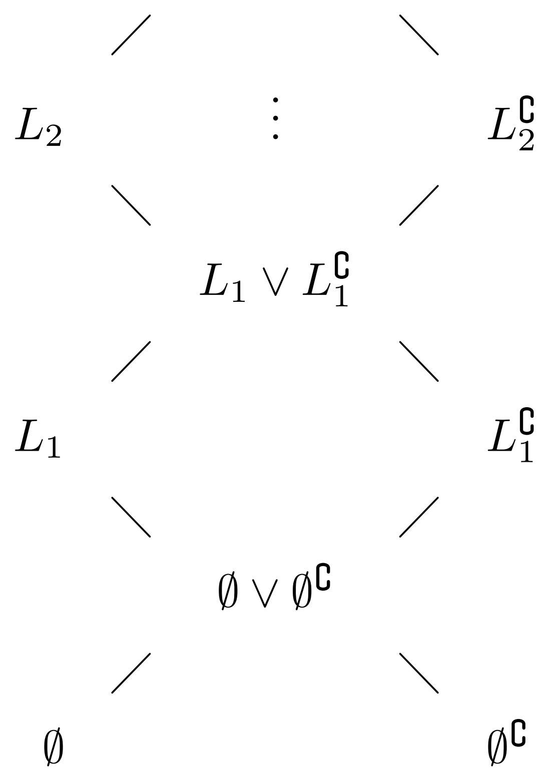
Under determinacy:

- $\leq$has antichains of lenght at most 2

- $\leq$ has no infinite descending chain:

$$L_0 > L_1 > L_2 > \ldots L_n > L_{n+1} > \ldots$$

Wadge hierarchy



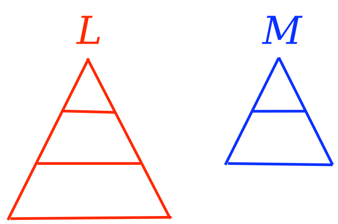
Wadge degree

- degree $(\emptyset)$=1
- degree (L)=sup$\{$degree$(M) : \ M < L\}$

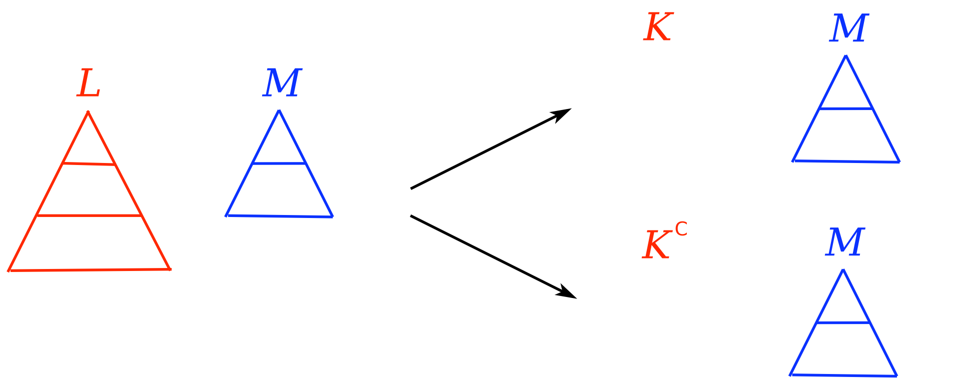
First results

- Measure of strength: height of hierarchy for $\Delta_2^{\mathbf{K}}$;
- part of this work derives from previous works on weak alternating tree automata, with Filip Murlak

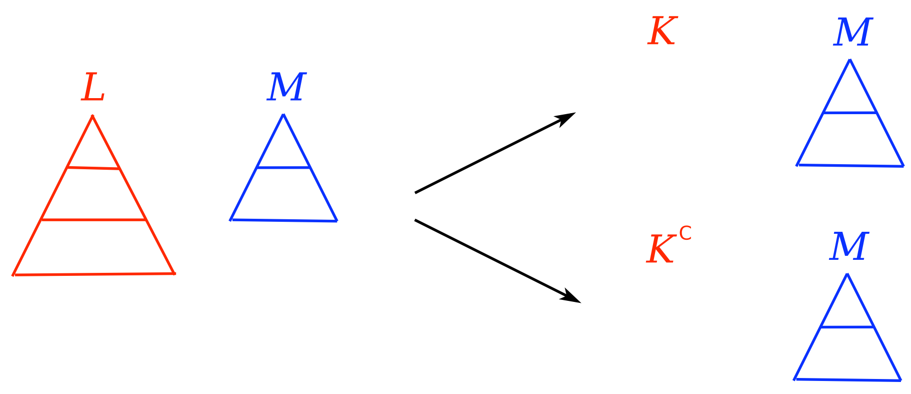
Addition



Addition

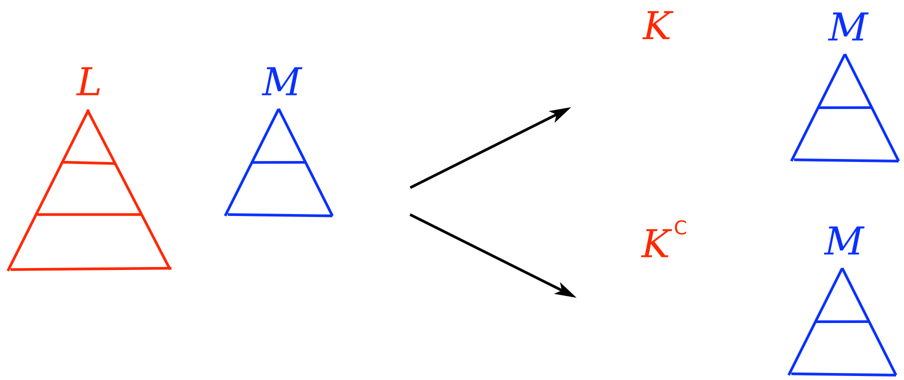


Addition



$L + K \ = \ L$ with the possibility to switch to K or $K^{\complement}$

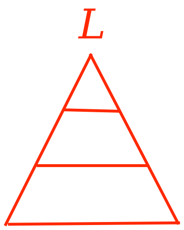
Addition



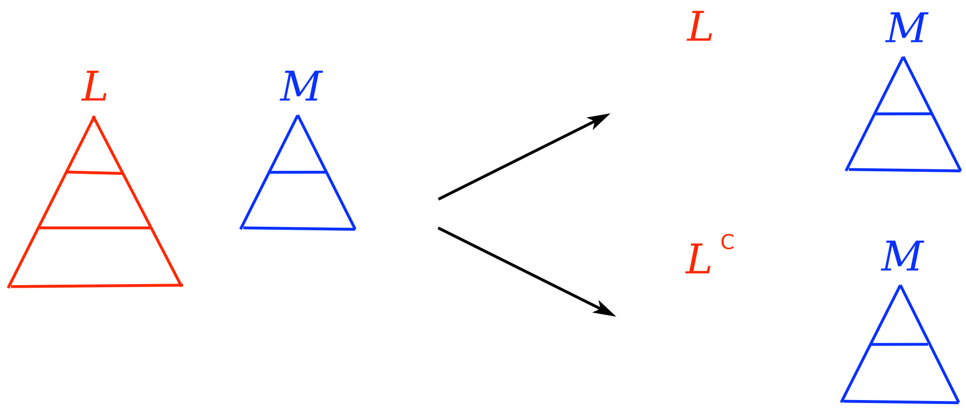
$L + K \ = \ L$ with the possibility to switch to K or $K^{\complement}$

$L, K \in \Delta_2^{\mathbf{K}} \implies L + K \in \Delta_2^{\mathbf{K}}$

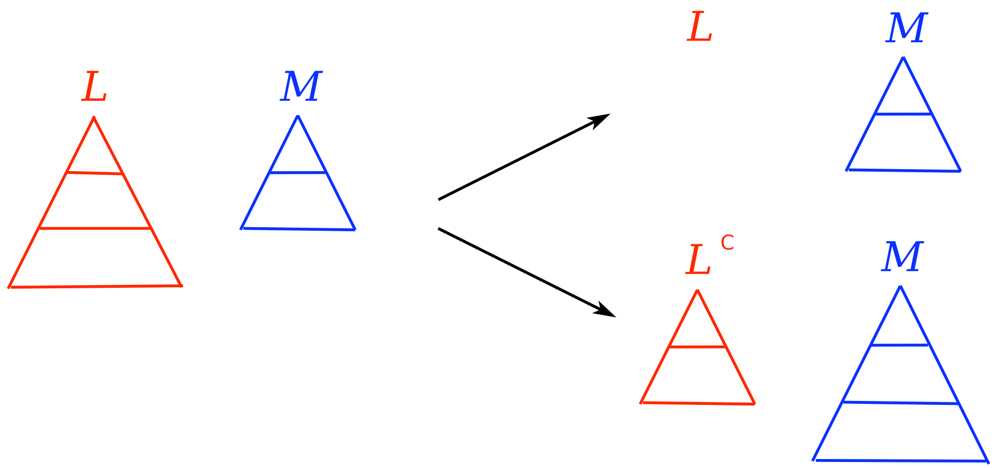
Multiplication



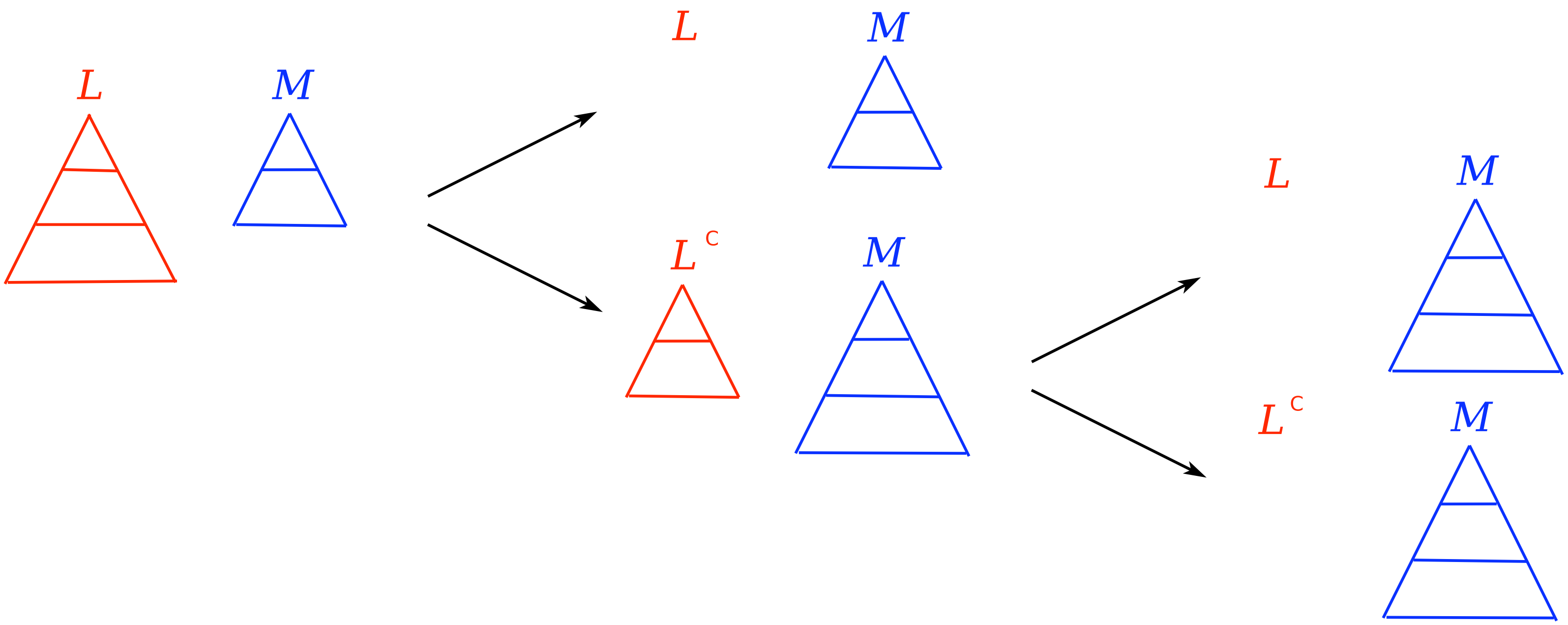
Multiplication



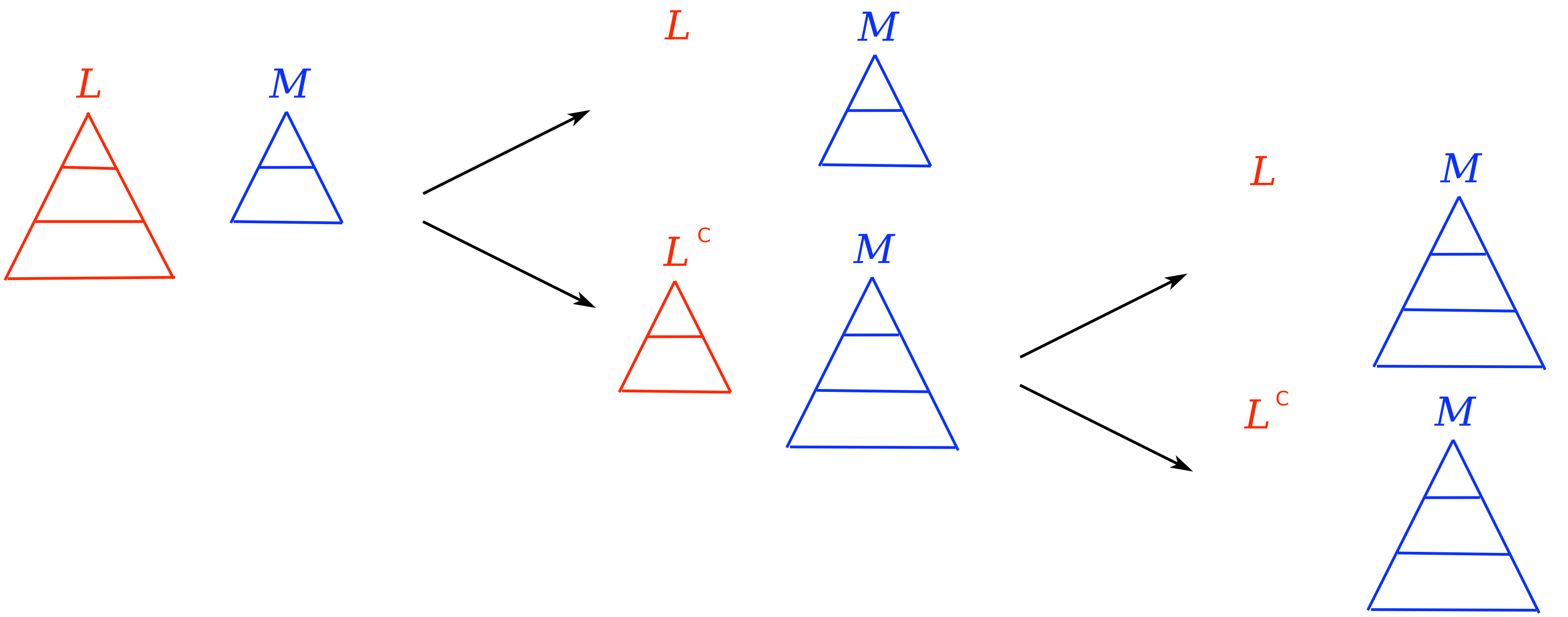
Multiplication



Multiplication

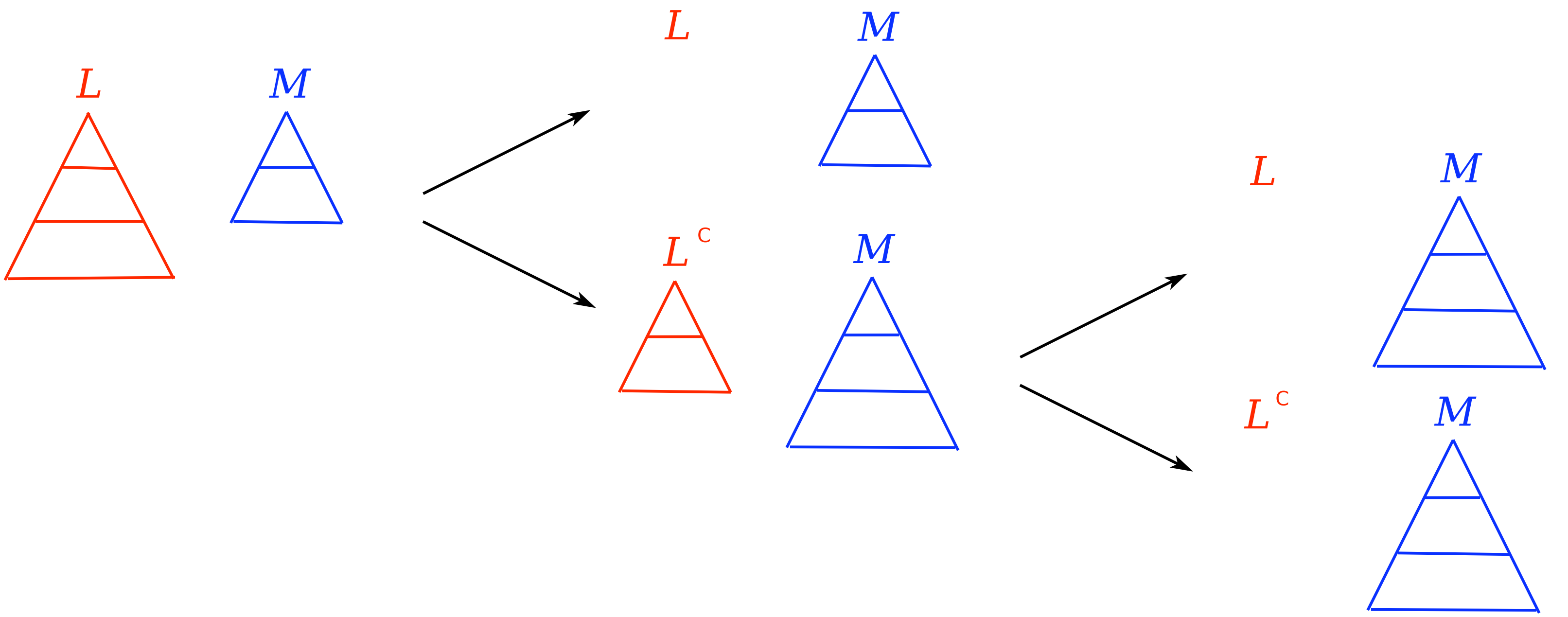


Multiplication



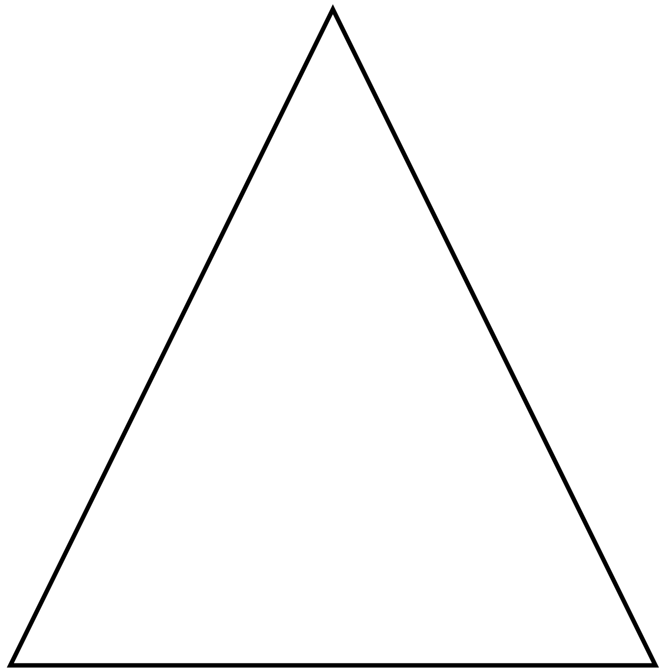
$L \in \Delta_2^{\mathbf{K}} \implies L \cdot k \in \Delta_2^{\mathbf{K}}$ for each k

Multiplication

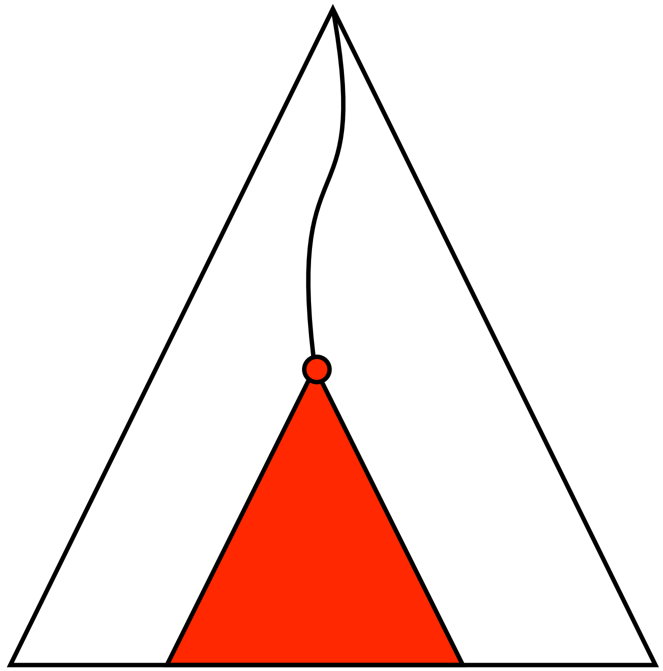


$L \in \Delta_2^{\mathbf{K}} \implies L \cdot k \in \Delta_2^{\mathbf{K}}$ for each k $L \in \Delta_2^{\mathbf{K}} \implies L \cdot \omega \in \Delta_2^{\mathbf{K}}$

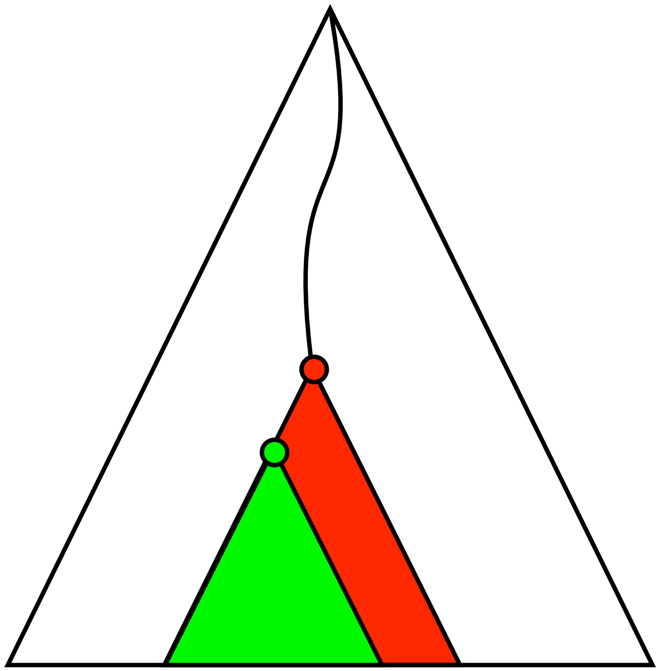
Exponentiation



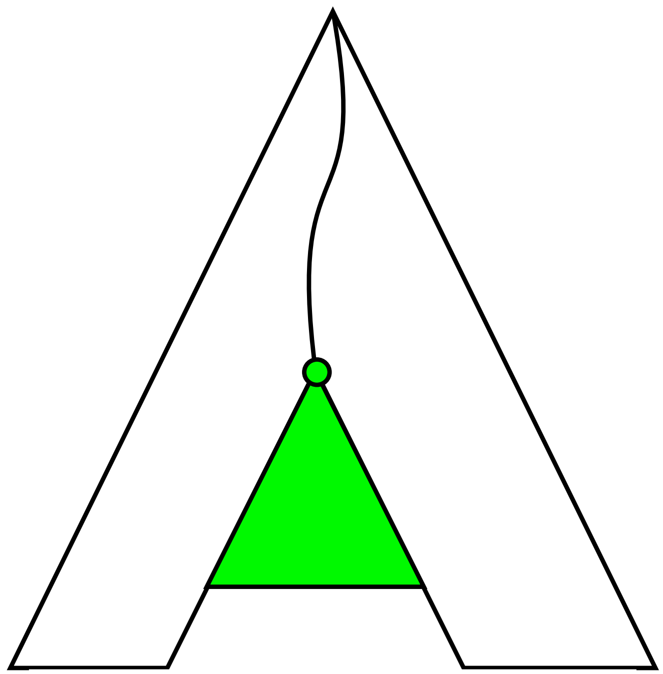
Exponentiation



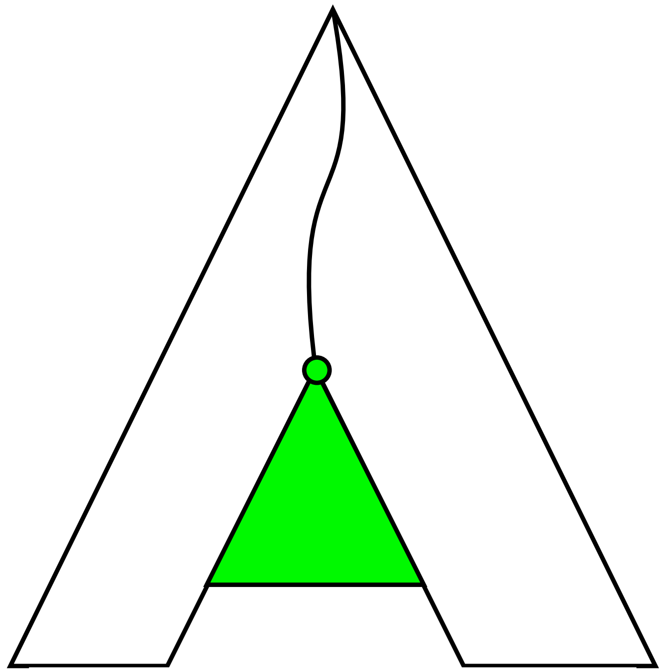
Exponentiation



Exponentiation



Exponentiation



$L \in \Delta_2^{\mathbf{K}} \implies \Omega^L \in \Delta_2^{\mathbf{K}}$

$\Delta_2^{\mathbf{K}}$ sets

The hierarchy of $\Delta_2^{\mathbf{K}}$ sets is closed under

addition $\alpha, \beta \rightsquigarrow \alpha + \beta$,

multiplication $\alpha \rightsquigarrow \alpha \cdot \omega$,

exponentiation $\alpha \rightsquigarrow \omega_1^{\alpha}$,

and hence has the height at least $\varepsilon_0 = \mu x.\omega^x = \omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}}$.

$\Delta^{\mathbf{K}}_2$ sets

The hierarchy of $\Delta^{\mathbf{K}}_2$ sets is closed under

addition

multiplication $< \omega^\omega$,

exponentiation of base ω_1,

$$\bigcup_{n\in\mathbb{N}} \mathbf{\Sigma}^0_n$$

The hierarchy of $\bigcup_{n\in\mathbb{N}} \mathbf{\Sigma}^0_n$ is closed under

addition

multiplication $< \omega_1$,

exponentiation of base ω_1,

$$\Delta^{\mathbf{K}}_2 \subset \bigcup_{n\in\mathbb{N}} \mathbf{\Sigma}^0_n$$

$\Delta_2^{\mathbf{K}}$ sets

Conjecture: The height **is** ε_0

$\Delta^{\mathbf{K}}_2$ sets

We know that there is no $\Delta^{\mathbf{K}}_2$ set between levels ω^ω and ω_1.

Future Work

Prove conjecture right

Decidability for $\Delta_2^{\mathbf{K}}$

Generate the whole hierarchy

Decidability

Prove strong relations between Wadge degrees and fad

$\boldsymbol{\Delta}_{\mathbf{2}}^{\mathbf{1}}$ Wadge hierarchy