Logic and Computation in Finitely-Presentable Infinite Structures

Lecture 4: Büchi and Rabin theorems. Tree interpretable structures.

Valentin Goranko and Sasha Rubin

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Büchi automata on ω-words

A Büchi automaton is a tuple $< \Sigma, S, \iota, F, \Delta >$, where:

- Σ is the alphabet
- S is a finite set of *states*
- $\iota \in S$ is the *initial state*
- $F \subseteq S$ is a set of *accepting states*
- $\Delta \subseteq S \times \Sigma \times S$ is a *transition function*.

Input: infinite word $\alpha \in \Sigma^\omega$.

Run: infinite word $\rho \in S^\omega$ so that $\rho[0] = \iota$ and for every $n \in \mathbb{N}$, $\Delta(\rho[n], \alpha[n], \rho[n+1])$.

Acceptance condition for Büchi automaton

$\mathtt{inf}(\rho)$: the set of states that occur infinitely often in ρ.

A word α is *accepted* by the automaton if some run ρ on α satisfies $\mathtt{inf}(\rho) \cap F \neq 0$ (Büchi condition).

EXAMPLES Fix $\Sigma = \{0, 1\}$.

- The set of ω-words with infinitely many 1s.
- The set of ω-words with finitely many 1s.
- The set of ω-words with an even number of 1s.

Regular relations and convolution $\otimes$

An ω-language is called (ω-)regular if it is exactly the set of ω-words accepted by some Büchi automaton.

We want to define ω-regular relations on $(\Sigma^\omega)^r$ that have good closure properties.

The idea: convert R into $\otimes R \subset (\Sigma^r)^\omega$ and then run an automaton with larger alphabet (Σ^r).

- $\otimes(0^\omega, 1^\omega) := \binom{0}{1}^\omega$.
- $\otimes(01^\omega, 1^\omega) := \binom{0}{1}\binom{1}{1}^\omega$.

Convolution $\otimes : (\Sigma^\omega)^r \to (\Sigma^r)^\omega$ where $\otimes(\alpha)[i] = (\alpha_1[i], \cdots, \alpha_r[i])$ for every $i \in \mathbb{N}$. Define $\otimes(R)$ as $\{\otimes(\alpha) \mid \alpha \in R\}$.

A relation R of ω-words is called (ω)-regular if $\otimes(R)$ is regular.

Examples of ω-regular relations

Think of $\alpha \in \underline{2}^\omega$ as the binary representation of a real number r_α in $[0, 1]$.

- $\{(\alpha, \beta) \mid r_\alpha \leq r_\beta\}$.
- $\{\alpha \mid r_\alpha \leq q\}$, for a fixed rational $q \in [0, 1]$.
- $\{(\alpha, \beta, \gamma) \mid r_\alpha + r_\beta = r_\gamma \mod 1\}$.

Büchi automata and closure properties

PROPOSITION. Regular relations are effectively closed under Boolean operations and FO logical operations.

That is, if $\otimes(R), \otimes(S)$ are ω-regular, then so is $\otimes(T)$ where T is one of:

- $R \cap S$, $R \cup S$, $R \setminus S$;
- the projection of R;

Moreover, an automaton for $\otimes(T)$ can be constructed from automata for $\otimes(R)$ and $\otimes(S)$.

(Intersection requires a little reflection, but complementation requires more work.)

Using automata to prove decidability: from sets to characteristic strings

THEOREM [Büchi, 1962] The MSO theory of $(\mathbb{N}, \mathrm{s})$ is decidable.

PROOF SKETCH: A set $A \subseteq \mathbb{N}$ can be viewed as an infinite string $\mathtt{str}(A) \in \underline{2}^\omega$, namely its characteristic string.
$[\mathtt{str}(A)[i] = 1$ if and only if $i \in A]$.

- $\mathtt{str}(\{n\}) = 0^n 1 0^\omega$;
- $\mathtt{str}(2\mathbb{N}) = (10)^\omega$;
- A is infinite if and only if $\mathtt{str}(A)$ has infinitely many 1s.

- $\{\mathtt{str}(X) \mid X \text{ is finite}\}$.
- $\{(\mathtt{str}(X), \mathtt{str}(Y)) \mid X \subseteq Y\}$

Using automata to prove decidability: from relations to characteristic strings

IDEA: With every formula $\phi(X_1, \cdots, X_r)$ of the MSO theory of $(\mathbb{N}, \mathsf{s})$, we associate the language consisting of ω-words $\otimes((\mathtt{str}(A_1), \cdots, \mathtt{str}(A_r)))$ for which $(\mathbb{N}, \mathsf{s}) \models \phi(A_1, \cdots, A_r)$.

- $\otimes\mathtt{str}\{(n, n+1) \mid n \in \mathbb{N}\}$ is the language $\binom{0}{0}^{\star}\binom{1}{0}\binom{0}{1}\binom{0}{0}^{\omega}$
- $\otimes\mathtt{str}\{(A, B) \mid A \subseteq B\}$ is the language $(\binom{0}{0}, \binom{0}{1}, \binom{1}{1})^{\omega}$.

What do the MSO-definable relations look like under this coding ?

Every MSO-definable relation in $(\mathbb{N}, \mathrm{s})$ is regular

PROPOSITION. Let $\Phi(X_1, \cdots, X_r)$ be an MSO-formula of $(\mathbb{N}, \mathrm{s})$, and Φ defines the relation R.
Then $\otimes\mathtt{str}(R) \subseteq (\{0,1\}^r)^\omega$ is regular.

PROOF SKETCH Induction on complexity of Φ.

- Atomic relations: $\otimes\mathtt{str}(\mathrm{s})$ and $\otimes\mathtt{str}(\subseteq)$ are regular.
- Boolean connectives: regular relations closed under Boolean operations.
- What about MSO quantification?
  $\exists X \Psi(X, \overline{Y})$ corresponds to projection $\exists\alpha B_\Psi(\alpha, \overline{\beta})$.

MSO of $(\mathbb{N}, s)$ is decidable

COROLLARY: The MSO-theory of $(\mathbb{N}, s)$ is decidable.

PROOF SKETCH: Translate model checking of formulae in $(\mathbb{N}, s)$ to decidable problems on regular languages.

For instance, truth of $(\mathbb{N}, s) \models \exists X \phi(X)$ is done by checking whether the Büchi automaton B_ϕ corresponding to ϕ accepts the empty language or not.

Converse: ω-regular relations in $(\mathbb{N}, \mathrm{s})$ are MSO definable

THEOREM. An ω-regular relation in $(\mathbb{N}, \mathrm{s})$ can be regarded as a characteristic relation of a set of tuples $(A_1, \cdots, A_r)$, where $A_i \subseteq \mathbb{N}$. [So $\mathtt{str}(R) \subseteq (\underline{2}^\omega)^r$ and $\otimes\mathtt{str}(R) \subseteq (\underline{2}^r)^\omega$]

The following are equivalent:

1. R is MSO-definable in $(\mathbb{N}, \mathrm{s})$.
2. $\otimes\mathtt{str}(R)$ is regular.
3. $\mathtt{str}(R)$ is FO-definable in $\mathcal{W}^\omega(\underline{2})$

Recall $\mathcal{W}^\omega(\underline{2}) := (\underline{2}^\omega; (\mathrm{s}_a)_{a \in \underline{2}}, \preceq_{\mathtt{prefix}}, =_{\mathrm{len}})$.

On infinite words, identify u with $\overrightarrow{u} := u10^\omega$.

- $\mathrm{s}_a : \overrightarrow{\Sigma^\star} \to \overrightarrow{\Sigma^\star}$ defined by $\mathrm{s}_a(\overrightarrow{u}) := \overrightarrow{ua}$,
- $\overrightarrow{u} \preceq_{\mathtt{prefix}} \beta$:if u is a prefix of β,
- $=_{\mathrm{len}} (\overrightarrow{u}, \overrightarrow{v})$:if $|u| = |v|$.

ω-regular relations in $(\mathbb{N}, s)$ are FO-definable in $\mathcal{W}^\omega(\underline{2})$

Recall: $[\mathtt{str}(R) \subseteq (\underline{2}^\omega)^r$ and $\otimes\mathtt{str}(R) \subseteq (\underline{2}^r)^\omega]$

Proposition. If $\otimes\mathtt{str}(R)$ is regular then $\mathtt{str}(R)$ is FO-definable in $\mathcal{W}^\omega(\underline{2})$.

$(r = 1)$ For Büchi automaton $B =< \Sigma, \{1, \cdots, |S|\}, \iota, F, \Delta >$, write a formula $\Phi_B(x)$ so that $\alpha \in L(B)$ iff $\mathcal{W}^\omega(\underline{2}) \models \Phi_B(\alpha)$:

$$\exists \beta_1 \cdots \exists \beta_{|S|} \in \underline{2}^\omega \, (\forall \overrightarrow{\pi}) \, START \wedge TRANS \wedge FINAL$$

$$START : \beta_\iota[\overrightarrow{\lambda}] = 1 \wedge \bigwedge_{s \neq s' \in S} \neg[\beta_s[\overrightarrow{\pi}] = \beta_{s'}[\overrightarrow{\pi}] = 1]$$

$$TRANS : \bigvee_{\Delta(s,a,s')} [\beta_s[\overrightarrow{\pi}] = 1 \wedge \alpha[\overrightarrow{\pi}] = a \wedge \beta_{s'}[\overrightarrow{\mathtt{s}_1\pi}] = 1]$$

$$FINAL : (\exists \overrightarrow{\delta})[|\overrightarrow{\pi}| < |\overrightarrow{\delta}| \wedge \bigvee_{s \in F} \beta_s[\overrightarrow{\delta}] = 1]$$

FO-definable relations in $\mathcal{W}^\omega(\underline{2})$ are MSO-definable in $(\mathbb{N}, \mathsf{s})$

[R set of tuples $(X_1, \cdots X_r)$ and $\mathtt{str}(R) \subseteq (\underline{2}^\omega)^r$]

PROPOSITION. If $\mathtt{str}(R)$ is FO-definable in $\mathcal{W}^\omega(\underline{2})$, then R is MSO-definable in $(\mathbb{N}, \mathsf{s})$.

PROOF: Follows from the result that $\mathcal{W}^\omega(\underline{2})$ is set interpretable in $(\mathbb{N}, \mathsf{s})$ (with co-ordinate map $X \mapsto \mathtt{str}(X)$) and the TRANSFER OF DEFINABILITY theorem.

Putting everything together

THEOREM. Let R be a set of tuples $(A_1, \cdots, A_r)$, where $A_i \subseteq \mathbb{N}$. The following are equivalent:

1. R is MSO-definable in $(\mathbb{N}, \mathsf{s})$.
2. $\otimes\mathtt{str}(R)$ is regular.
3. $\mathtt{str}(R)$ is FO-definable in $\mathcal{W}^\omega(\underline{2})$

Tree automata and Rabin's Theorem

An ω-tree automaton is a tuple $< \Sigma, S, \iota, \mathcal{F}, \Delta >$, where:

- Σ is the input alphabet,
- S is a finite set of *states*,
- $\iota \in S$ is the *initial state*,
- $\mathcal{F}$ consists of subsets of S, the *accepting sets*,
- $\Delta \subseteq S \times \Sigma \times S^2$ is a *transition function*.

A *run* on input ω-tree $T \in \mathit{tree}(\Sigma)$ is an ω-tree $R \in \mathit{tree}(S)$ satisfying for every $w \in \underline{2}^\star$,

$$\Delta(R(w), T(w), R(w0), R(w1)).$$

A run R is *accepting* if every path $\alpha \in \underline{2}^\omega$ of R satisfies $\mathtt{inf}(\alpha) \in \mathcal{F}$ (Müller condition).

Rabin's Theorem

CODING: A set $A \subseteq \underline{2}^\star$ can be viewed as an ω-tree $\mathtt{tree}(A) : \underline{2}^\star \to \underline{2}$ satisfying

$$\mathtt{tree}(A)(w) = 1 \iff w \in A.$$

For every MSO-formula $\phi(X_1, \cdots, X_r)$ of $(\{0,1\}^\star, \mathsf{s}_0, \mathsf{s}_1)$, associate the language consisting of $\oplus(\mathtt{tree}(A_1), \cdots, \mathtt{tree}(A_r))$ for which $(\{0,1\}^\star, \mathsf{s}_0, \mathsf{s}_1) \models \phi(A_1, \cdots, A_r)$. Here $\oplus$ maps a tuple of trees $(t_1, \cdots, t_r)$ to a single tree $\oplus(t)$ over alphabet 2^r; namely for every $w \in 2^\star$, $\oplus(t)(w) := (t_1(w), \cdots, t_r(w))$.

Rabin's theorem

THEOREM[Rabin, 1969, Blumensath & Grädel 1999/2000] Let R be a set of tuples $(A_1, \cdots, A_r)$, where $A_i \subseteq \underline{2}^\star$. The following are equivalent:

1. R is MSO-definable in $(\{0,1\}^\star, \mathsf{s}_0, \mathsf{s}_1)$.
2. $\oplus\mathtt{tree}(R)$ is recognised by an ω-tree-automaton.
3. $\mathtt{tree}(R)$ is FO definable in $\mathcal{T}^\omega(\underline{2})$.

The hard part: ω-tree automata are closed under complementation.

COROLLARY: The MSO of the infinite binary tree

$$\mathcal{T}_2 = (\{0,1\}^\star, \mathsf{s}_0, \mathsf{s}_1)$$

is decidable.

Summary of definability and decidability results

1. $\text{WMSO}(\mathbb{N}, \mathsf{s}) \leftrightarrows$ automata on finite words $\leftrightarrows \text{FO}(\mathcal{W}(\underline{2}))$.
   Büchi 1960, Elgot 1961, Trahtenbrot 1962 – Eilenberg, Elgot, Shepherdson 1969.
2. $\text{MSO}(\mathbb{N}, \mathsf{s}) \leftrightarrows$ automata on infinite words $\leftrightarrows \text{FO}(\mathcal{W}^{\omega}(\underline{2}))$
   Büchi 1962 - Blumensath, Grädel 1999/2000.
3. $\text{WMSO}(\{0,1\}^{\star}, \mathsf{s}_0, \mathsf{s}_1) \leftrightarrows$ automata on finite trees $\leftrightarrows$ $\text{FO}(\mathcal{T}(\underline{2}))$.
   Thatcher, Wright 1968, Doner 1970 - Blumensath, Grädel 1999/2000.
4. $\text{MSO}(\{0,1\}^{\star}, \mathsf{s}_0, \mathsf{s}_1) \leftrightarrows$ automata on infinite trees $\leftrightarrows$ $\text{FO}(\mathcal{T}^{\omega}(\underline{2}))$
   Rabin 1969 - Blumensath, Grädel 1999/2000.

Tree Interpretable Structures

A structure $\mathcal{A}$ is tree-interpretable if $\mathcal{A} \leq_{\mathsf{MSO}} \mathcal{T}_2$.

THEOREM [DECIDABILITY OF MSO]:
Every tree-interpretable structure has a decidable MSO-theory.

Tree Interpretable Structures: examples

Examples (Rabin, 1969):

- $(\mathbb{Q}, \leq)$.
- $\mathcal{T}_m = (\{0, 1, \ldots, m-1\}^\star, \mathsf{s}_0, \mathsf{s}_1, \ldots, \mathsf{s}_{m-1})$, for any $m \geq 2$.

PROOF SKETCH: Code (injectively) the string $i_1 \ldots i_k$ in $\mathcal{T}_m$ by the string $1^{i_1+1}0 \ldots 1^{i_k+1}0$ in $\mathcal{T}_2$.
Thus, $\Delta^{\mathcal{T}_m} = \{10 + 110 + \cdots + 1^m 0\}^*$.

The formula $\Delta(x)$:

$$\forall X[x \in X \wedge \forall y((y10 \in X \vee \ldots \vee y1^m0 \in X) \rightarrow y \in X) \rightarrow \epsilon \in X].$$

The formula $\Phi_{\mathsf{s}_j}(x, y)$:

$$\Phi_{\mathsf{s}_j}(x, y) := \exists x_1 \ldots \exists x_j[\mathsf{s}_1(x, x_1) \wedge \ldots \wedge \mathsf{s}_1(x_{j-1}, x_j) \wedge \mathsf{s}_0(x_j, y)].$$

- $\mathcal{T}_\omega$.

Pushdown Automata

A pushdown automaton:

$$\mathcal{P} = (A, Q, \Gamma, q_0, Z_0, \Delta),$$

where:

- A is input alphabet;
- Q is a finite set of control states;
- Γ is stack alphabet;
- $q_0 \in Q$ is initial state;
- $Z_0 \in \Gamma$ is initial stack symbol;
- $\Delta \subseteq Q \times A \times \Gamma \times \Gamma^* \times Q$ is transition function.

A configuration of $\mathcal{P}$: **(current control state, stack content)**.
Formally, a word in $Q\Gamma^*$.
Acceptance: by empty stack.
Pushdown automata accept precisely the *context-free languages*.

Pushdown Graphs

Given a pushdown automaton $\mathcal{P} = (A, Q, \Gamma, q_0, Z_0, \Delta)$, its configuration /transition/ graph is

$$G_{\mathcal{P}} = (V, \{E_a\}_{a \in A}),$$

where V is the set of all configurations of $\mathcal{P}$ reachable from the initial state $q_0 Z_0$, and each E_a is a (transition) relation defined by:

$$p\gamma w \; E_a \; q\zeta w \text{ iff } (p, \gamma) \vdash^a (q, \zeta) \in \Delta;$$

Pushdown graph: the transition (configuration) graph of a pushdown automaton.

Thus, a pushdown graph can be infinite, but every vertex has *finite* in- and out- degrees.

Examples: $(\mathbb{N}, \mathrm{s})$; any $\mathcal{T}_m$, for $m \geq 2$.

Muller and Schupp's characterization of pushdown graphs

Consider a transition graph $G = (V, \{E_a\}_{a \in A})$ with a designated 'initial' vertex v_0.
V_n: the set of vertices in G reachable from v_0 by a non-oriented path of length $\leq n$.
G_n: the subgraph of G induced by the set of vertices $V \backslash V_n$. The vertices in $V_{n+1} \backslash V_n$: *boundary vertices* of G_n.
The ends of the graph G: the connected (via non-oriented paths) components of G_n, for $n \leq 0$.

Theorem [Muller and Schupp, 1985]: A transition graph of bounded degree is a pushdown graph iff the number of distinct isomorphism types of its ends is finite.

Example: $\mathcal{T}_2$. Non-example: the infinite grid $\mathbb{N} \times \mathbb{N}$.

Muller and Schupp's theorem

THEOREM [MULLER AND SCHUPP, 1985]: Every pushdown graph is tree-interpretable.

COROLLARY: Every pushdown graph has a decidable MSO-theory.

Prefix Recognizable Graphs

Σ: a fixed finite alphabet.

Prefix-rewriting system on a language $V \subseteq \Sigma^*$: a finite set of rewrite rules of the type $U_1 \rightarrow_a U_2$ where U_1, U_2 are regular subsets of V.

There is a transition of type a from a word v_1 to a word v_2 iff there is a rule $U_1 \rightarrow_a U_2$ and words $v \in V$, $u_1 \in U_1, u_2 \in U_2$ such that $v_1 = u_1 v$ and $v_2 = u_2 v$.

A (transition) graph $G = (V, \{E_a\}_{a \in A})$ is prefix-recognizable over Σ if $V \subseteq \Sigma^*$ is regular and the transitions in G are determined by some prefix-rewriting system on V. Thus, for each $a \in A$,

$$E_a = \bigcup_{i=1}^{m} (U_i^1 \times U_i^2) V_i = \bigcup_{i=1}^{m} \{(u_1 v, u_2 v) \mid v \in V_i, u_1 \in U_i^1, u_2 \in U_i^2\}$$

for some regular languages $V_i, U_i^1, U_i^2, i = 1, \ldots, m$.

Prefix Recognizable Graphs: examples

Every pushdown graph is prefix recognizable: every transition can be written as a rewriting rule on the configuration graph.

In fact, the pushdown graphs are (up to isomorphism) precisely the prefix-recognizable graphs with finite in- and out-degrees.

But, there are prefix-recognizable graphs in with infinite out-degree.

Prefix Recognizable Graphs Are Tree Interpretable

THEOREM [CAUCAL, 1996]: Every prefix recognizable graph is tree-interpretable.

PROOF SKETCH. Let $G = (V, \{E_a\}_{a \in A})$ be a prefix recognizable graph, where $V \subseteq \Sigma^*$, and let m be the size of Σ. Then we show that $G \leq_{\text{MSO}} \mathcal{T}_m$:

Let $\mathcal{A}$ be a finite automaton recognizing V. Then the formula $\Delta(x)$ says 'x is recognized by $\mathcal{A}$' by simulating the runs of $\mathcal{A}$. *Exercise: construct* $\Delta(x)$.

Fix a transition rule $\rho : U_1 \rightarrow_a U_2$ where U_1, U_2 are regular subsets of V, recognized by some finite automata $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively. We then define an MSO formula $\Phi_\rho(x, y)$ meaning:
'The word y is obtainable from the word x by applying the rule ρ'.
Exercise: construct $\Phi_\rho(x, y)$.

COROLLARY: Every prefix recognizable graph has a decidable MSO-theory.

Pushdown Graphs with ϵ-transitions

Extend the definition of a pushdown automaton $\mathcal{P} = (A, Q, \Gamma, q_0, Z_0, \Delta)$, by allowing ϵ-transitions, i.e. re-defining the transition relation as:

$$\Delta \subseteq Q \times (A \cup \{\epsilon\}) \times \Gamma \times \Gamma^* \times Q$$

The configuration graph of a pushdown automaton with ϵ-transitions is obtained after factoring out the configurations not occurring in any non-ϵ transitions.

Pushdown graphs with ϵ-transitions recognize the same languages (viz., context free), but the configuration graph of such an automaton can have vertices with infinite in- and out-degrees.

Rational Restrictions and Inverse Rational Mappings

Let $G = (V, \{E_a\}_{a \in A})$ where $V \subseteq \Sigma^*$ is a regular language. For any $C \subseteq \Sigma^*$, we denote by $G|_C$ the subgraph of G induced by $V \cap C$. When C is regular, $G|_C$ is called a rational restriction of G.

Let $\overleftarrow{A}$ be a disjoint copy of A. For every $a \in A$ we define $E_{\overleftarrow{a}} := (E_a)^{-1}$, for any $\overleftarrow{a} \in \overleftarrow{A}$.

Given a set of labels B we consider a mapping h which assigns to every $b \in B$ a language $h(b) \subseteq (A \cup \overleftarrow{A})^*$. The words in $h(b)$ represent paths in G expanded with $\{E_{\overleftarrow{a}}\}_{\overleftarrow{a} \in \overleftarrow{A}}$.

Now, we define the inverse rational mapping (or, inverse substitution) $h^{-1}(G)$ to be the graph $G = (V, \{E'_b\}_{b \in B})$ where $uE'_b v$ iff there is a path in G from u to v labeled by a word in $h(b)$.

Inverse substitutions are particular case of MSO interpretations.

Caucal's original definition of prefix-recognizable graphs was: those obtained as inverse rational substitutions from $\mathcal{T}_2$, followed by

Characterization of the Tree Interpretable Graphs

THEOREM [CAUCAL'1996, BARTHELMANN'1997, STIRLING'2000, BLUMENSATH'2001]: For any graph G the following are equivalent, up to isomorphism:

1. G is tree-interpretable.
2. G is prefix-recognizable.
3. G is a rational restriction of the configuration graph of a pushdown automaton with ϵ-transitions.
4. G is a rational restriction of the inverse image of $\mathcal{T}_2$ under a rational mapping h.
5. G is VR-equational (Courcelle).