Limit Computable Functions and Subsets on Metric Spaces

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joint work with
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Survey

- Limit Computable Functions
- Limit Computable Real Numbers
- Limit Decidable Sets
- Limit Computable Sets are Exactly the Effective Δ^0_2 Sets
- Limit Located Sets
- Conclusion

Limit Computable Functions

Definition 1 A function $f :\subseteq \Sigma^\omega \to \Sigma^\omega$ is called *limit computable* if there is a two-way output Turing machine which eventually transfers the input $p \in \Sigma^\omega$ to its corresponding output $F(p) \in \Sigma^\omega$ such that any position on the output tape is changed only a finite number of times.

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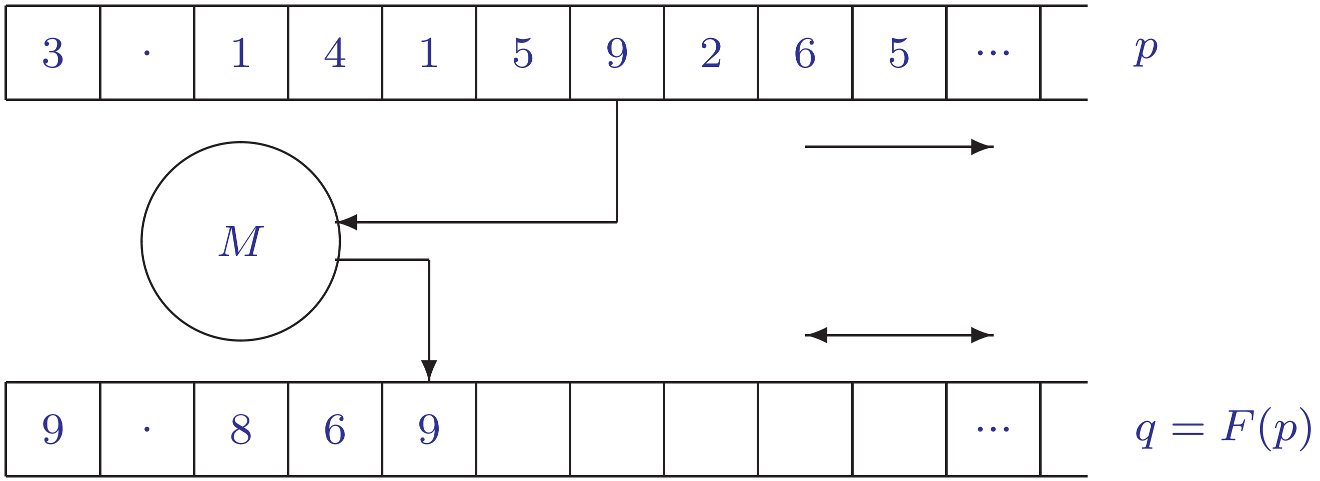

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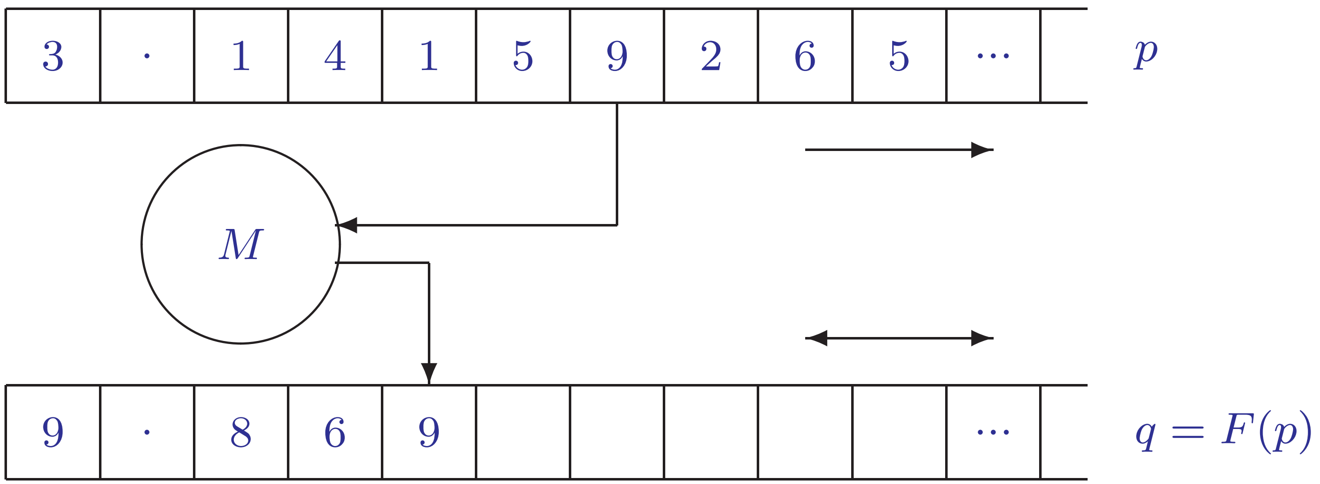

Remark 2 *Limit computable functions are not closed under composition.*

Limit Computable Functions

Theorem 3 (Composition Theorem) *Let f be limit computable and g computable then $f \circ g$ and $g \circ f$ are both limit computable.*

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Definition 5 For represented spaces (M, γ_1) and (N, γ_2), a function $f :\subseteq M \rightarrow N$ is said to be (γ_1, γ_2)-limit computable if it has a (γ_1, γ_2)-limit realizer $F :\subseteq \Sigma^\omega \rightarrow \Sigma^\omega$.

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Definition 6 (Derivative of a Representation) Let δ_X be a representation of the set X. Then we define the "jump" of δ_X, δ'_X, by

$$\delta'_X \langle p_0, p_1, ...\rangle = \delta_X \left(\lim_{n \to \infty} p_n \right)$$

Limit Computable Functions

Theorem 7 *Let* (X, δ_X) *and* (Y, δ_Y) *be represented spaces and let* $f :\subseteq X \to Y$ *be a function. Then the following are equivalent:*

1. $f :\subseteq X \to Y$ *is limit computable with respect to* (δ_X, δ_Y)*,*
2. $f :\subseteq X \to Y$ *has a limit computable realizer with respect to* (δ_X, δ_Y)*,*
3. $f :\subseteq X \to Y$ *has a* $\mathbf{\Sigma}_2^0$*–computable realizer with respect to* (δ_X, δ_Y)*,*
4. $f :\subseteq X \to Y$ *has a computable realizer with respect to* (δ_X, δ'_Y)*.*

Limit Computable Functions

Definition 8 (Function Space Representations) For the sets $[X \to Y]$, $[X \to' Y]$; via representation η and η' of the continuous and the $\mathbf{\Sigma}_2^0$-measurable partial string functions with $\mathbf{\Pi}_2^0$ and $\mathbf{\Pi}_3^0$–domains respectively:

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$$[\delta_X \to \delta_Y](q) = f :\iff \eta(q) = F \text{ and } \delta_Y F(p) = f\delta_X(p) \text{ for all } p \in \mathrm{dom}(f\delta_X).$$

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Corollary 10 *And* $[\delta_X \to \delta_Y]' \equiv [\delta_X \to \delta'_Y]|^{[X \to Y]}$.

Limit Computable Real Numbers

Definition 11 (Limit computable point) Let (X, δ) be a represented space. Then $x \in X$ is called limit computable, if there is a computable $p \in \Sigma^\omega$ such that $\delta'(p) = x$.

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Proposition 12 (Limit computable reals) *For a real number $x \in \mathbb{R}$ the following are equivalent:*

1. x *is limit computable,*
2. x *has a limit computable ρ–name,*
3. x *is computable with respect to the naive Cauchy representation,*
4. $x = \sup_{n\in\mathbb{N}} \inf_{k\in\mathbb{N}} r_{nk} = \inf_{n\in\mathbb{N}} \sup_{k\in\mathbb{N}} s_{nk}$ *for two computable double sequences* $(r_{nk})_{n,k\in\mathbb{N}}$ *and* $(s_{nk})_{n,k\in\mathbb{N}}$ *of real (or rational) numbers.*

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Lemma 13 *Uniformly (Weihrauch and Zheng)*
$\rho' \equiv \rho_{><} \sqcap \rho_{<>} \equiv \rho'_{<} \sqcap \rho'_{>}$.

Limit Decidable Sets

Definition 14 Let (X, δ_X) be a represented space. A subset $A \subseteq X$ is called *limit decidable* if the characteristic function

$$\chi_A : X \to \mathbb{R} : x \mapsto \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

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Definition 15 (The representation $\delta_{\mathcal{L}(X)}$ of $\mathcal{L}(X)$) Let (X, δ_X) be a represented space. We define a representation of $\mathcal{L}(X)$ by

$$\delta_{\mathcal{L}(X)}(p) = A \;:\Longleftrightarrow\; \chi_A = [\delta_X \to \rho_{\mathrm{Cn}}](p)$$

Limit Decidable Sets

Lemma 16 (Union, Intersection and Set Complementation) *For* $A, B \in \mathcal{L}(X)$

- *the operation* $A \mapsto X \backslash A$ *on* $\mathcal{L}(X)$ *where* $A \subseteq X$ *is* $(\delta_{\mathcal{L}(X)}, \delta_{\mathcal{L}(X)})$*-computable,*
- *the operations* $(A, B) \mapsto A \cup B$ *and* $(A, B) \mapsto A \cap B$ *on* $\mathcal{L}(X)$ *where* $A, B \subseteq X$ *are* $(\delta_{\mathcal{L}(X)}, \delta_{\mathcal{L}(X)}, \delta_{\mathcal{L}(X)})$*-computable.*

Limit Decidable Sets

Definition 17 (The representation $\delta_{\boldsymbol{\Delta}^0_2}$ of Δ^0_2)

1. We define $\delta_{\boldsymbol{\Pi}^0_2}$, a representation of the class of all G_δ-sets. A $\delta_{\boldsymbol{\Pi}^0_2}$ name p for a G_δ-set A encodes some open sets that countably intersect to A:

$$\delta_{\boldsymbol{\Pi}^0_2}\langle p_0, p_1, p_2, ...\rangle = A :\Longleftrightarrow A = \bigcap_{i\in\mathbb{N}} \delta_{\Sigma^0_1}(p_i)$$

where $\delta_{\Sigma^0_1}(p_i) = \bigcup_{\langle i,j\rangle\in\text{range}(p)} B(\alpha(i), \bar{j})$

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$$\delta_{\mathbf{\Sigma}^0_2}(p) = A :\Longleftrightarrow \delta_{\mathbf{\Pi}^0_2}(p) = X \backslash A$$

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3. Let $\delta_{\mathbf{\Delta}^0_2} := \delta_{\mathbf{\Pi}^0_2} \wedge \delta_{\mathbf{\Sigma}^0_2}$.

Limit Decidable Sets

Theorem 18 (Effectively Δ_2^0-sets are Exactly the Limit Decidable Ones)
For any computable metric space X *we have* $\delta_{\mathbf{\Delta}_2^0} \equiv \delta_{\mathcal{L}(X)}$.

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Theorem 18 (Effectively Δ^0_2-sets are Exactly the Limit Decidable Ones) *For any computable metric space* X *we have* $\delta_{\mathbf{\Delta}^0_2} \equiv \delta_{\mathcal{L}(X)}$.

Proof. "$\leq$"

- Let $\delta_{\mathbf{\Delta}^0_2} \langle p, q \rangle = A$ which means $\delta_{\mathbf{\Pi}^0_2}(p) = \bigcap U_i = A = \delta_{\mathbf{\Sigma}^0_2}(q) = X \setminus \bigcap V_i$.
- We describe a limit machine LM which uses this info together with the ρ name for some $x \in X$ to limit evaluate χ_A.

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- We describe a limit machine LM which uses this info together with the ρ name for some $x \in X$ to limit evaluate χ_A.
- LM writes 0 on the first cell of the output tape (the initial assumption being $x \notin A$)

  At stage $i = 0, 1, 2, \ldots$

  1. LM checks if x is in U_i, if so the machine changes the result from 0 to 1.
  2. LM checks if x is in V_i, if so the machine changes the result from 1 to 0.

Limit Decidable Sets

- It follows that there is a computable function f such that $\chi_A \circ \rho(r) = \rho_{\mathrm{Cn}} \circ f(\langle p, q\rangle, r)$.
- Using transposition, we obtain a computable function g such that $\chi_A = [\rho^n \to \rho_{\mathrm{Cn}}](g \langle p, q\rangle)$.
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"$\geq$"

Consequence of the F_σ–measurability of χ_A. $\square$

Limit Located (Closed) Sets

Definition 19 (Representation $\psi^{'}_{\text{dist}}$ of $\mathcal{A}^*(\mathbb{R}^n)$) The distance function for a closed set A on x is defined as $d_A(x) = \inf_{y \in A} ||x - y||$. We call a closed set A *limit located* if its distance function d_A is limit computable. We define a representation $\psi^{'}_{\text{dist}}$ as follows:

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Theorem 20 *Restricted to $\mathcal{A}^*(\mathbb{R}^n)$; limit decidable, r.e. closed, co-r.e. closed and recursively closed sets are all limit located.*

Proof. A set is called r.e. closed or co-r.e. closed if its distance function is upper or lower semi-computable, respectively. Since $\rho_< \equiv_{\mathrm{lim}} \rho_>$ hold, the claim follows. □

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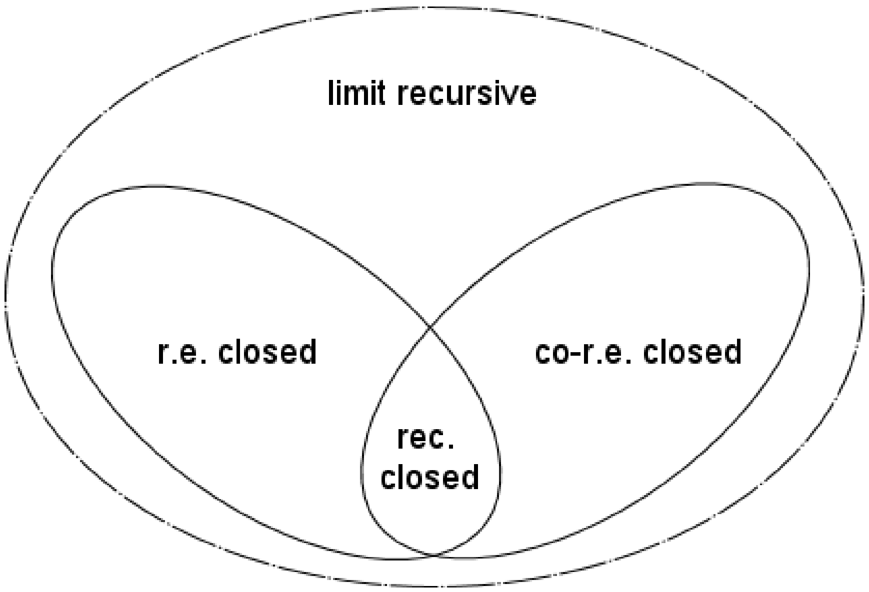

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Lemma 21 *For any computable metric space, $\mathbf{EQU}$ is co-r.e. closed.*

Lemma 22 *For any computable metric space, $\mathbf{EQU}$ is decidable in the limit.*

Example 23 (An r.e.-closed set which is not effectively $\mathcal{L}(X)$) *Hemmerling showed that the set*

$$A_\infty = \{n + \tfrac{1}{3+k} \mid \mathrm{card}(W_n) > k \wedge k \in \mathbb{N}\} \cup \{n \mid W_n \textit{ is infinite}\}$$

is r.e.–closed but is not in $\mathcal{L}(X)$.

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Theorem 24 (Positive information does not suffice to compute χ_A of A)
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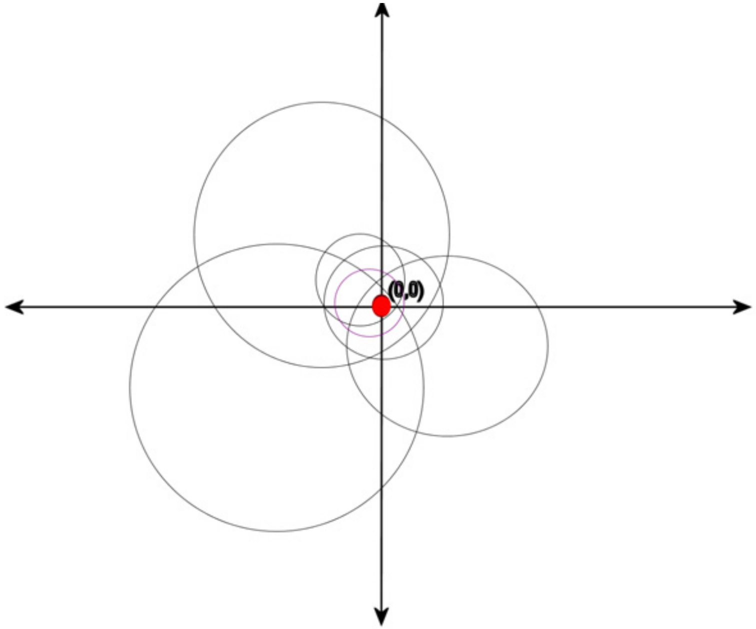
Proof.

- Assume $\mathrm{id} : \mathcal{A} \to \mathcal{A}, A \mapsto A$ is $(\psi_+, \delta_{\mathcal{L}(X)})$–continuous hence has a continuous realizer F.
- Start with the origin with positive information $A_0 = \{0\} = \psi_+(p)$.
- We evaluate $\chi_{A_0}(0) = [\rho \to \rho_{\mathrm{Cn}}](p_0)(0)$, then there is a prefix $p_{0,\leq n_0}$, on which the answer is yes.
- We take all the balls enumerated by the this string and choose a rational point in them excluding zero, let this new set be A_1 with ψ_+-name $p_1 = p_{0,\leq n_0}q$ for some sequence q.
- We then evaluate $\chi_{A_1}(0) = [\rho \to \rho_{\mathrm{Cn}}](p_1)(0)$ then there is some prefix $p_{1,\leq n_1} = p_{0,\leq n_0}w$ for which the answer is yes.
- We create $A_2 = A_1 \cup \{0\}$ with name $p_2 = p_{0,\leq n_0}p_{1,\leq n_1}q$

Limit Located (Closed) Sets



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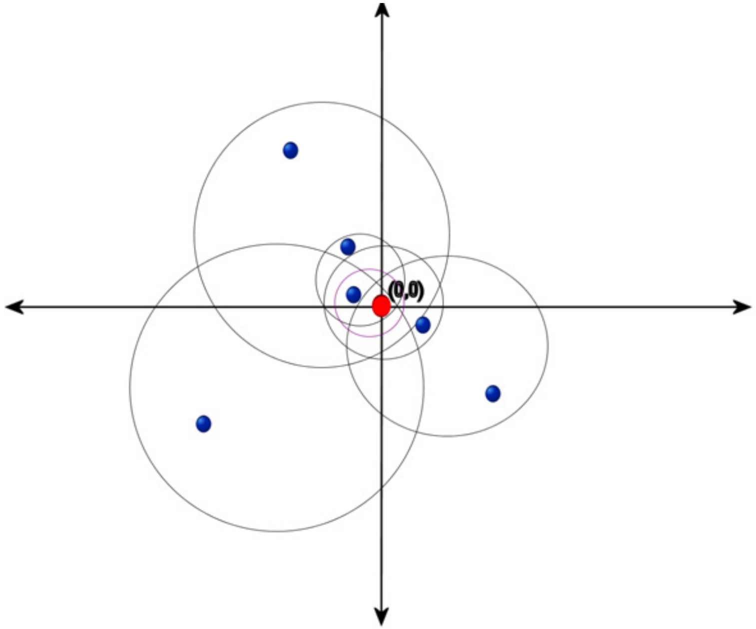

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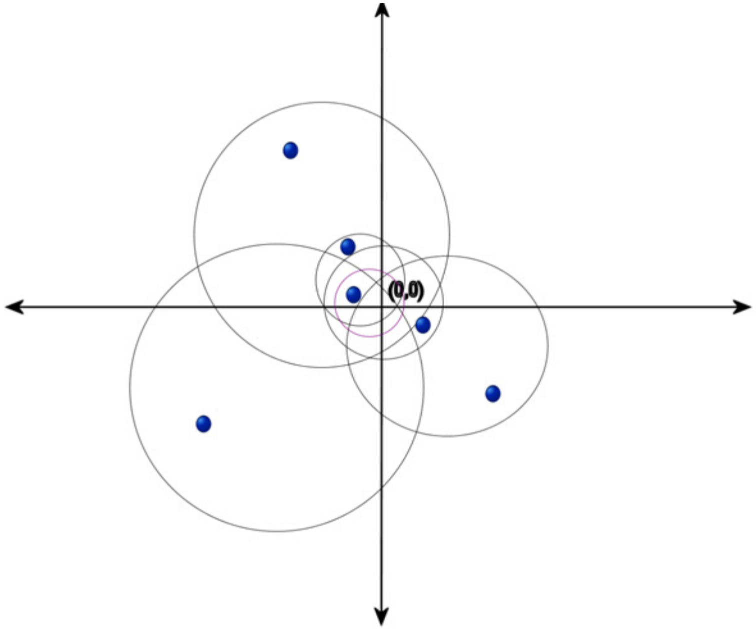

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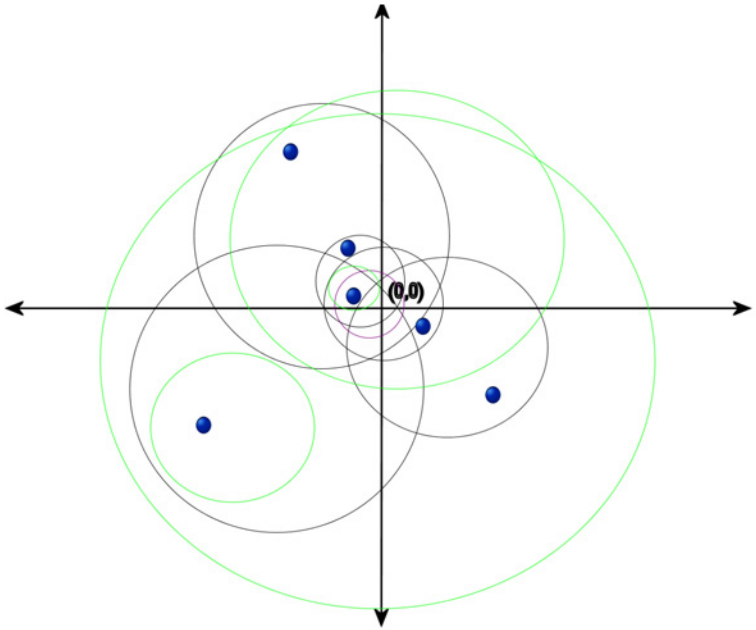

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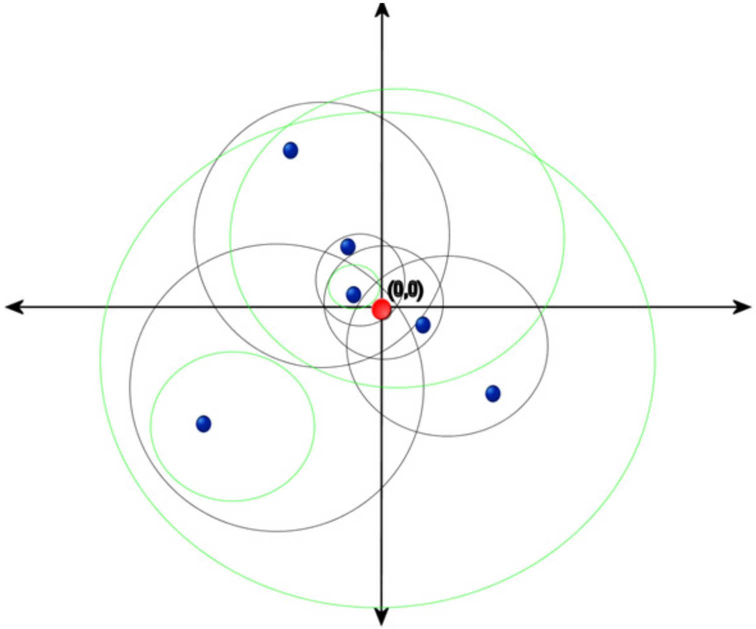

- We carry on inductively to define A_∞, an r.e. closed set with positive name $p_\infty = p_{0,\leq n_0} p_{1,\leq n_1} \cdots$.
- But it is quite clear from the behavoir of the machine on input "0", at a computational level that F, the realizer is not computable.
- One could along similar reasoning style show that the function F does not have a monotone inducer hence it is not continuous.

Limit Located (Closed) Sets

□

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3. A possible future direction: topological properties of general metric spaces.
4. Thank you!