Computational Issues in the Theory of Bounded Analytic Functions

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Outline

1. Background from analysis
2. Quick introduction to TTE
3. The basic computability results
   - Zero sets
   - Blaschke products
   - Inner functions
4. Possible research directions
5. References

The class $H^\infty(\mathbb{D})$

Let $\mathbb{D} = \{z \in \mathbb{C} \ : \ |z| < 1\}$.

- $H^\infty(\mathbb{D})$ is the set of all bounded analytic functions $f : \mathbb{D} \to \mathbb{C}$.
- For $f \in H^\infty(\mathbb{D})$, let

$$\| f \|_\infty = \sup\{|f(z)| \ : \ z \in \mathbb{D}\}.$$

- $H^\infty(\mathbb{D})$ is a Banach space under $\| \ \|_\infty$.

Inner functions

Definition

$u \in H^\infty(\mathbb{D})$ is *inner* if $\lim_{z \to z_0} |u(z)| = 1$ for almost all $z_0 \in \partial\mathbb{D}$.

- Inner functions "generate" $H^\infty(\mathbb{D})$.
- Two important classes of inner functions:
  - Singular functions
  - Blaschke products

Definition

A function $s \in H^\infty(\mathbb{D})$ is *singular* if there is a finite positive Borel measure on $\partial\mathbb{D}$, μ, that is singular with respect to Lebesgue measure and such that

$$s(z) = \exp\left\{-\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} d\mu(e^{it})\right\}$$

Theorem

If s is singular, then s is inner, s has no zeros, and $s(0)$ is a positive real number.

Definition

Let $A = \{a_n\}_{n<l}$ be a (possibly infinite) sequence of points in $\mathbb{D} - \{0\}$. The product

$$B_{A,k}(z) =_{df} z^k \prod_{n<l} \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a_n} z}$$

is called a *Blaschke product*. We abbreviate $B_{A,0}$ with B_A.

Definition

Let $A = \{a_n\}_{n<l}$ be a (possibly infinite) sequence of points in $\mathbb{D} - \{0\}$. The sum

$$\Sigma_A =_{df} \sum_{n<l} (1 - |a_n|)$$

is called the *Blaschke sum* of A. The inequality $\Sigma_A < \infty$. is called the *Blaschke condition*.

Theorem

Let $A = \{a_n\}_{n<l}$ be a sequence of points in $\mathbb{D} - \{0\}$.

1. *If A satisfies the Blaschke condition, then $B_{A,k}$ is an inner function.*
2. *If A satisfies the Blaschke condition, then the terms of A are precisely the zeros of A repeated according to multiplicity.*
3. *If A does not satisfy the Blaschke condition, then $B_A \equiv 0$.*

Theorem

If u is an inner function, and if $a_0, a_1, \ldots$ are the zeros of u (repeated according to multiplicity), then $\{a_n\}_n$ is a Blaschke sequence.

Definition

If u is inner, then let $\Sigma_u = \sum_n (1 - |a_n|)$ where $a_0, a_1, \ldots$ are the zeros of u repeated according to multiplicity.

Theorem

(Factorization of Inner Functions) *If u is an inner function, then there exist unique λ_u, b_u, s_u such that $u = \lambda_u b_u s_u$, $\lambda_u \in \partial\mathbb{D}$, b_u is a Blaschke product, and s_u is a singular function.*

Capacity

Let $E \subseteq \mathbb{D}$ be compact. We make the following definitions.

- $V^E(z_1, \ldots, z_n) = \prod_{i<k} |z_i - z_k|$.
- $V_n^E = \max\{V^E(z_1, \ldots, z_n) \mid z_1, \ldots, z_n \in E \text{ distinct}\}$.
- $d_n^E = \sqrt[\binom{n}{2}]{V_n^E}$

It follows that $d_n^E \geq d_{n+1}^E$. We define the *capacity* of E to be $\lim_{n\to\infty} d_n$.

If $F \subseteq \mathbb{D}$, then F has *zero capacity* if all of its compact subsets have zero capacity.

Theorem

Every zero-capacity set has measure zero.

However, the Cantor middle-third set has *positive* capacity.

For $a, z \in \overline{\mathbb{D}}$ with $|a| < 1$, let

$$M_a(z) = \frac{z - a}{1 - \overline{a}z}.$$

Theorem

(Frostman's Theorem) *Let u be a non-constant inner function. Then, $M_a \circ u$ is a unit multiple of a Blaschke product for all $a \in \mathbb{D}$ except in a set of capacity zero.*

The set of values of a for which $M_a \circ u$ is not a unit multiple of a Blaschke product is called the *exception set of u*.

As $a \to 0$, $\| M_a \circ u - u \|_\infty \to 0$.

Corollary

If u is a non-constant inner function, and if $\epsilon > 0$, then there is a unit multiple of a Blaschke product B such that $\| u - B \|_\infty < \epsilon$.

Naming systems Fix a finite alphabet Σ with $0, 1 \in \Sigma$.

Definition

A *notation for M* is a surjection $\nu :\subseteq \Sigma^* \to M$.

Definition

A *representation of M* is a surjection $\delta :\subseteq \Sigma^\omega \to M$.

Representations and notations are called *naming systems*.
If $\gamma(p) = x$, where γ is a representation or a notation, then p is a γ-*name of x*.
$x \in M$ is γ-*computable* if it has a computable γ-name.

Type-two machines and computable functions

Definition

A *type-two machine* is a Turing machine with one or more infinite read-only input tape(s) and an infinite write-only output tape.

Details and a precise definition can be found in Weihrauch's book.

Definition

Suppose δ_i is a naming system for M_i, $i = 0, \ldots, n+1$ and $f :\subseteq M_0 \times \ldots \times M_n \rightarrow M_{n+1}$. f is $(\delta_0, \ldots, \delta_{n+1})$-*computable* if there is a type-two machine with n input tapes such that whenever $(x_0, \ldots, x_n) \in dom(f)$ and a δ_i-name of x_i is written on the i-th input tape for $i = 0, \ldots, n$, a δ_{n+1}-name of $f(x_0, \ldots, x_n)$ is written on the output tape.

Details can be found in Weihrauch's book.

Naming systems to be used here

- ρ^2 for $\mathbb{C}$. A ρ^2-name of $z \in \mathbb{C}$ lists all open rational rectangles that contain z.
- $\theta_<$ for set of open subsets of $\mathbb{C}$. A $\theta_<$-name of U lists all closed rational rectangles contained in U.
- ψ for set of closed subsets of $\mathbb{C}$. A ψ-name of E lists :
  1. All open rational rectangles that intersect E.
  2. All closed rational rectangles that do not intersect E.

- δ_{open} for set of continuous $f :\subseteq \mathbb{C} \to \mathbb{C}$ with open domain. This representation is defined by:

  $$\delta_{open}(\langle p, q \rangle) = f \Leftrightarrow \eta_p^{\omega\omega} \ (\rho^2, \rho^2)\text{-realizes } f \wedge \theta_<(q) = dom(f).$$

  So, a δ_{open}-name of f consists of a $\theta_<$-name of its domain and a code of an oracle type-two machine that computes f with respect to ρ^2.

- For $\mathbb{C}^{\leq\omega}$, we use the naming system δ_{seq} defined by

$$\delta_{seq}(p) = A \Leftrightarrow \psi_<(p) = graph(A).$$

A δ_{seq}-name of A lists all open rational rectangles that intersect the graph of A.

$A \mapsto lh(A)$ is not computable with respect to this representation.

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Let f range over analytic functions only.

Zero sets

Theorem

(McNicholl, 2007) *$f \mapsto f^{-1}[\{0\}]$ is not computable.*

Theorem

(Matheson, McNicholl, PAMS 2008) *$f \not\equiv 0 \mapsto f^{-1}[\{0\}]$ is computable.*

Outline

Theorem

(Matheson, McNicholl, PAMS 2008) *$B_A \rightrightarrows A$ is computable.*

Theorem

(Matheson, McNicholl, PAMS 2008) *There is a computable sequence A such that B_A is not computable.*

In other words, merely knowing the Blaschke sequence is not enough to compute the Blaschke product.

Theorem

(Matheson, McNicholl PAMS 2008) *The map*

$$(A, \Sigma_A) \mapsto B_A$$

is computable.

In other words, if you know a Blaschke sequence and its Blaschke sum, then you can compute the Blaschke product.

Theorem

(McNicholl, 2007) *The map* $(A, B_A) \mapsto \sum_A$ *is computable. In fact,* $(A, B_A(0)) \mapsto \sum_A$ *is computable.*

In other words, once you know a Blaschke sequence, in order to compute the Blaschke product you have to know the Blaschke sum (or an equivalent piece of information).

Corollary

(**McNicholl 2007)** *Suppose A is computable. If B_A maps computable complex numbers to computable complex numbers, then B_A is computable.*

This is not the case for power series!

Theorem

(**Caldwell, Pour-El 1975)** *There is a computable sequence $\{a_n\}_{n=0}^{\infty}$ such that $z \mapsto \sum_{n=0}^{\infty} a_n z^n$ is computable on closed disks but not on the entire plane.*

Outline

Estimation

Theorem

Computable Frostman Theorem (McNicholl, 2007) *There is a computable multivalued function:* $(u, \epsilon) \rightrightarrows B \ni \| u - B \|_\infty < \epsilon$.

This theorem shows we can *effectively* estimate inner functions by unit multiples of Blaschke products.

Orders of zeros Throughout the rest of these slides, u ranges over inner functions.
Let $ord(f, z_0)$ be the order of z_0 as a zero of f (defined to be 0 if $f(z_0) \neq 0$).

Theorem

(McNicholl, 2007) *$u \mapsto ord(u, 0)$ is not computable.*

Theorem

(McNicholl, 2007) *$(u, \Sigma_u) \mapsto ord(u, 0)$ is computable.*

Theorem

There is a computable Blaschke sequence A such that Σ_A (and hence B_A) is computable but $z \mapsto \mathrm{ord}(B_A, z)$ is not.

Factorization

Theorem

(McNicholl, 2007) *The map* $u \mapsto (\lambda_u, b_u, s_u)$ *is not computable.*

In other words, merely knowing an inner function is not enough to compute its factorization.

Theorem

(McNicholl, 2007) *The map* $(u, \sum_u) \mapsto (\lambda_u, b_u, s_u)$ *is computable.*

In other words, for inner functions, knowing the Blaschke sum of *u* is *sufficient* for computing the factorization of *u*.

Theorem

(McNicholl, 2007) *The map* $(u, b_u) \mapsto \sum_u$ *is computable.*

In other words, to compute the factorization of u, knowledge of the Blaschke sum of u (or equivalent information) is *necessary*.

Singular functions

$$s(z) = \exp\left\{-\int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(e^{it})\right\}$$

Investigate computability-theoretic aspects of singular functions. *e.g.*

- Is $s \mapsto \mu$ "computable"?
- If μ and $s(0)$ are computable, does it follow that s is computable?

Capacity Investigate computability-theoretic aspects of capacity. For example:

- Is there a computable closed set whose capacity is a non-computable real?
- If yes, what are the possible Turing degrees? (Any *c.e.* degree?)

Factorization

- Is there a computable inner function u with a non-computable factorization?
- What is the topology of $\{(u, \Sigma_u) \mid u \text{ inner}\}$? What are its connected components?

Exception sets Let $\delta_{F\sigma}(\langle p_0, p_1, \ldots \rangle) = F$ if

$$F = \bigcup_i \psi(p_i).$$

- If u is computable, does it follow that the exception set of u is $\delta_{F\sigma}$-computable?
- Is $u \mapsto$ (exception set of u) computable?

Random Blaschke products Investigate Blaschke sequences $\{a_n\}_{n=0}^{\infty}$ such that $\{|a_n|\}_{n=0}^{\infty}$ is computable but $\{arg(a_n)\}_{n=0}^{\infty}$ is "random". (*c.f.* Cochran, "Random Blaschke Products", TAMS, 1991.)

- Is such a product interpolating?
- A such a product in the little Bloch space?

Frostman condition

Definition

If $A = \{a_n\}_{n=0}^{\infty}$ is a sequence, then define

$$\sigma(A) = \sup_{\zeta \in \partial \mathbb{D}} \sum_{n=0}^{\infty} \frac{1 - |a_n|^2}{|\zeta - a_n|}.$$

The *Frostman condition* is: $\sigma(A) < \infty$.

Every Frostman sequence is a Blaschke sequence.
The Frostman condition “spaces out” the terms of A.
Are there interesting computability aspects of Frostman sequences? *e.g.* Is there a computable Frostman sequence such that Σ_A is computable but $\sigma(A)$ is not?

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