Computable Elementary Topology

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joint work with

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Contents

So far Computable Analysis mainly from the concrete to the abstract:
computable points, sets, functions
first for for $\mathbb{R}$, then $\mathbb{R}^n$, then metric spaces, ...

This talk: top-down approach to computable topology.

– Basic computability
– Computability on Σ^ω
– induced computability/continuity
– Computable T_0 spaces
– Natural representations of points, open sets, quasi-compact sets, partial continuous functions
– Some results
– Computable Hausdorff, T_3, T_4 spaces
– Furthere results

Basic Computability:

- Turing machines, computable functions on numbers $\mathbb{N}$, on words Σ^*
- transfer to countable sets M:
  numbering $\nu :\subseteq \mathbb{N} \to M$ or notation $\nu :\subseteq \Sigma^* \to M$,
  p is a "name" of x, if $\nu(p) = x$,
  finite information suffices to identify $x \in M$
  Turing machines operate on names.
- for uncountable sets M:
  infinite information needed for identifying $x \in M$,

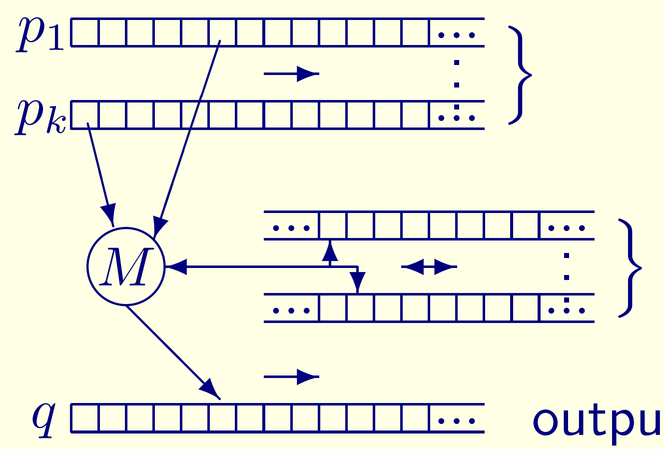
  Solution: use infinite sequences $p \in \Sigma^\omega$ as "names",

  representation approach, Type-2 Theory of Effectivity (TTE)

Computability on Σ^ω

Type-2 machine

computable function f_M



$f_M(p_1, \ldots, p_k) = q$, iff on input $(p_1, \ldots, p_k)$ the machine writes q.

Definition: V is r.e., iff V is the halting set of a Type-2 machine.

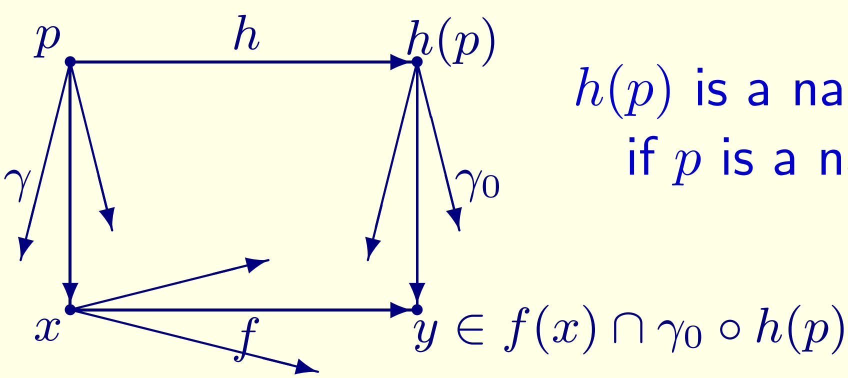
Theorem: f is continuous, iff $(\forall p)\, f(p) = g(q, p)$
for some computable g and some ("oracle") $q \in \Sigma^\omega$.

Induced Computability/Continuity

multi-representation $\delta : \Sigma^\omega \rightrightarrows M$

p is a "name" of $x \in M$, if $x \in \delta(p)$.

h realizes $f : M \rightrightarrows M_0$ w.r.t. multi-representations (γ, γ_0).



$h(p)$ is a name of some $y \in f(x)$
if p is a name of $x \in \mathrm{dom}(f)$

f is (γ, γ_0)-computable, iff it is realized by some computable h.

f is (γ, γ_0)-continuous, iff it is realized by some continuous h.

$\gamma \le \gamma_0$, $\gamma \equiv \gamma_0$: computable reducible, equivalent

$\gamma \le_t \gamma_0$, $\gamma \equiv_t \gamma_0$: continuous reducible, equivalent

Computable T_0-spaces

$\mathbf{X} = (X, \tau, \beta, \nu)$ such that

- (X, τ) is a second countable T_0-space,
- $\nu :\subseteq \Sigma^* \to \beta$ is a notation of a base β of τ with recursive domain,
- $U \neq \emptyset$ for $U \in \beta$,
- computable intersection

$$\nu(u) \cap \nu(v) = \bigcup \{\nu(w) \mid (u, v, w) \in E\} \quad \text{for some r.e. set } E\,.$$

Important Representations

- Points:

  Goal: "$x \in U$" is r.e. (for $x \in X$ and $U \in \beta$)

  Definition of $\delta :\subseteq \Sigma^\omega \to X$:

  $$\delta(p) = x \iff p \text{ is a list of all } w \in \Sigma^* \text{ such that } x \in \nu(u).$$

  Universal property of δ: For every representation δ' of X,

  $$\delta'(p) \in \nu(u) \quad \text{is r.e.} \iff \delta' \le \delta.$$

- Open sets:

  Observation: $(\forall V \subseteq X)\quad \{p \mid \delta(p) \in V\}$ is open, iff $V \in \tau$.

  Goal: "$x \in O$" is r.e. (for $x \in X$ and $O \in \tau$)

  Definition of $\theta :\subseteq \Sigma^\omega \to \tau$:

  $$\theta(p) = U \iff O = \bigcup\{\nu(w) \mid w \text{ is listed by } p\}$$

  Universal property of θ: For every representation θ' of τ,

  $$\delta(p) \subseteq \theta'(q) \quad \text{is r.e.} \iff \theta' \le \theta.$$

K is quasi-compact if every open cover of K has a finite subcover.

- Quasi-compact sets:

Observation:

$(\forall\, K \subseteq X) \quad \{p \mid K \subseteq \theta(p)\}$ is open, iff K is quasi-compact.

Goal: "$K \subseteq O$" is r.e. (for quasi-compact $K \subseteq X$ and $O \in \tau$)

Definition of $\kappa :\subseteq \Sigma^\omega \to \mathcal{K}$:

$\kappa(p) = K \iff p$ encodes a list of all finite base covers of K.

Universal property of κ: For every representation κ' of $\mathcal{K}$,

$$\kappa'(p) \subseteq \theta(q) \quad \text{is r.e.} \iff \kappa' \leq \kappa.$$

- Continuous partial functions $f :\subseteq X \to X'$:
  (for $\mathbf{X} = (X, \tau, \beta, \nu)$ and $\mathbf{X}' = (X', \tau', \beta', \nu')$ and representations δ, δ' of points)

  Goal: apply : $(f, x) \mapsto f(x)$ is computable.

  Definition of $\delta_{\to} : \Sigma^\omega \rightrightarrows \mathrm{PC}(X, X')$:

  $$f \in \delta_{\to}(p) \iff (\forall v \in \mathrm{dom}(\nu'))\, f^{-1}[\nu'(v)] = \mathrm{dom}(f) \cap \bigcup\{\nu(u) \mid (u, v) \text{ is listed by } p\}$$

  Universal property of $\delta_{\to}$:

  For every representation $\overline{\delta}$ of $\mathrm{PC}(X, X')$:

  apply is $(\overline{\delta}, \delta, \delta')$-computable $\iff \overline{\delta} \leq \delta_{\to}$.

- Relatively continuous partial functions $f :\subseteq X \to X'$:

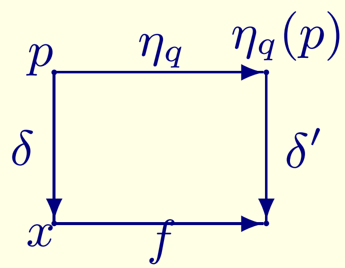
  (for $\mathbf{X} = (X, \tau, \beta, \nu)$ and $\mathbf{X}' = (X', \tau', \beta', \nu')$)

  Let $\eta : \Sigma^\omega \to F^{\omega\omega}$ be the canonical representation of the continuous partial functions $h :\subseteq \Sigma^\omega \to \Sigma^\omega$ with G_δ-domain.

  Definition of $[\delta \to_p \delta'] : \Sigma^\omega \to \mathrm{PC}(X, X')$:

  For (δ, δ')-continuous $f :\subseteq X \to X'$,

  $$f \in [\delta \to_p \delta'](q) \iff \eta_q \text{ is a } (\delta, \delta')\text{-realization of } f.$$



  Theorem: $[\delta \to_p \delta'] \equiv \delta_\to$

Some results:

With respect to the standard representations the following functions are computable:

- union and intersection on τ,
- countable union on τ,
- composition $(f, g) \mapsto f \circ g$ for continuous f and g,
- $(f, O) \mapsto f^{-1}[O]$ for continuous f and open O,
- $(f, K) \mapsto f[K]$ for continuous f and quasi-compact K.

A topological space (X, τ) is

- a Hausdorff space, if any two points can be separated by disjoint neighborhoods,
- a T_3-space, if every point x and every closed set not containing x can be separated by disjoint neighborhoods,
- a T_4-space, if any two disjoint closed sets can be separated by disjoint neighborhoods.

Consider a computable T_0-space $\mathbf{X} = (X, \tau, \beta, \nu)$ with representation

- δ of points,
- θ of open sets,
- ψ, $\psi(p) = X \setminus \theta(p)$, of closed sets,
- κ of quasi-compact sets.

Def. Computable Hausdorff Space:

The multi-function $(x,y) \rightrightarrows (U,V)$ is computable,
where $x, y \in X$, $U, V \in \beta$, $x \in U$, $y \in V$, $U \cap V = \emptyset$.

Def. Computable T_3 Space:

The multi-function $(x,A) \rightrightarrows (U,V)$ is computable,
where $x \in X$, A is closed, $U, V \in \tau$, $x \in U$, $A \subseteq V$, $U \cap V = \emptyset$.

Def. Computable T_4 Space:

The multi-function $(A,B) \rightrightarrows (U,V)$ is computable, where
A, B are closed, $U, V \in \tau$, $A \subseteq U$, $B \subseteq V$, $U \cap V = \emptyset$.

Theorem:

computable Hausdorff $\Longleftarrow$ computable T_3 $\Longleftrightarrow$ computable T_4

Let $\rho :\subseteq \Sigma^\omega \to \mathbb{R}$ be the standard representation of the real line.

Theorem: (computable Urysohn)
For every computable T_3 space
the multi-function $(A, B) \rightrightarrows f$ is $(\psi, \psi, [\delta \to \rho])$-computable,
where A, B are non-empty closed, $A \cap B = \emptyset$, $f : X \to [0; 1]$ is continuous, $f[A] = 0$, $f[B] = 1$

Theorem: (computable metrization)
For every computable T_3 space there is a metric $d : X \times X \to \mathbb{R}$ generating the topology that is (δ, δ, ρ)-computable.