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Does the isotropy of the CMB imply a homogeneous universe? Some generalized EGS theorems

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Abstract. We demonstrate that the high isotropy of the cosmic microwave background (CMB), combined with the Copernican principle, is not sufficient to prove homogeneity of the universe—in contrast to previous results on this subject. The crucial additional factor not included in earlier work is the acceleration of the fundamental observers. We find the complete class of irrotational perfect fluid spacetimes admitting an exactly isotropic radiation field for every fundamental observer and show that they are Friedmann–Lemaître–Robertson–Walker (FLRW) if and only if the acceleration is zero. While inhomogeneous in general, these spacetimes all possess three-dimensional symmetry groups, from which it follows that they also admit a thermodynamic interpretation.

In addition to perfect fluids models we also consider multi-component fluids containing non-interacting radiation, dust and a quintessential scalar field or cosmological constant in which the radiation is isotropic for the geodesic (dust) observers. It is shown that the non-acceleration of the fundamental observers forces these spacetimes to be FLRW.

While it is plausible that fundamental observers (galaxies) in the real universe follow geodesics, it is strictly necessary to determine this from local observations for the cosmological principle to be more than an assumption. We discuss how observations may be used to test this.

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1. Introduction

The high isotropy of the cosmic microwave background (CMB) is usually taken as strong evidence that the universe is homogeneous and isotropic, i.e. is well described by an FLRW model. The principle justification for this is an important theorem of Ehlers, Geren and Sachs (EGS, 1968) (based on earlier work by Tauber and Weinberg 1961), which states that if all observers in an expanding, dust universe measure an isotropic CMB then the universe is FLRW and the cosmological principle is valid. The importance of this theorem lies in the fact that it permits the homogeneity and isotropy of the universe to be deduced not from measurements of the actual isotropy of the universe about us, but only from measurements of the CMB, combined with the Copernican principle (that is, the assumption that all observers in the universe see the same degree of isotropy). The Copernican principle is often regarded as a powerful but untestable assumption in cosmology, although there are suggestions that it may be testable using the Sunyaev–Zeldovich effect, for example (Goodman 1995). Here we simply assume that the Copernican principle is valid and study the consequences of applying it to the observed high degree of isotropy of the CMB. That is, we examine spacetimes with an isotropic CMB for *all* observers.

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The EGS theorem has been generalized by Treciokas and Ellis (1971) to include an isotropic collision term. Ferrando *et al* (1992) find the general form of the energy–momentum tensor and Einstein's equations for spacetimes with an isotropic radiation field, and consider some special cases with anisotropic pressure. It has also been shown by Stoeger *et al* (1995) that the EGS theorem almost holds when applied to an almost isotropic radiation field.

There are counterexamples to the spirit of the EGS theorem (that is, when some of the assumptions are relaxed the result fails to hold). In particular, Ellis *et al* (1978) show that the result does not hold if the expansion is zero (which is obviously not relevant to cosmology), and Ferrando *et al* (1992) emphasize that homogeneity does not follow if there is anisotropic pressure in the energy–momentum tensor. Most importantly for the work presented here, though, is the result of Barrett and Clarkson (1999a), which shows that when the assumption of geodesic observers is relaxed there exist inhomogeneous perfect fluid (or scalar field) cosmologies with an isotropic CMB. (In fact, we show in this paper that the Stephani models considered in Barrett and Clarkson (1999a) are representatives of the only perfect fluid spacetimes that admit an isotropic CMB for all fundamental observers.) Nilsson *et al* (1999) provide a counterexample to the almost EGS result when the Weyl curvature is not negligible.

The basis of the EGS theorem is the Liouville equation for photons, which tells us that if a radiation field (i.e. a solution of the Liouville equation) exists such that for every observer on some timelike congruence the radiation field is isotropic, then that congruence is (parallel to) a conformal Killing vector (CKV). This may be expressed more formally as follows (Ehlers *et al* 1968, Ferrando *et al* 1992):

Theorem 1. *A spacetime will admit an isotropic radiation field if and only if it is conformal to a stationary spacetime, which happens if and only if there is a velocity field u^a satisfying*

$$\sigma_{ab} = 0, \tag{1}$$

$$\nabla_{[a}\big(\dot{u}_{b]} - \tfrac{1}{3}\theta u_{b]}\big) = 0, \tag{2}$$

where σ_{ab}, $\dot{u}^a$ and θ are the shear, acceleration and expansion of u^a, respectively.

(Then u^a is the velocity field relative to which the radiation is isotropic, and is parallel to the CKV.)

In the absence of some statement about the matter content of a spacetime, or further assumptions about the congruence u^a, theorem 1 is all that can be said. In a cosmological context the simplest, and most common, assumption is that the matter is dust (implying that u^a is geodesic), which leads to:

Theorem 2 (Ehlers, Geren and Sachs). *If the fundamental observers in a dust spacetime see an isotropic radiation field, then the spacetime is locally FLRW.*

Alternatively, we can simply assume that u^a is geodesic. The existence of an isotropic radiation field then ensures (for non-zero expansion) that the energy flux relative to u^a is zero. If the anisotropic stress tensor is zero at any instant (so that the energy–momentum tensor has perfect fluid form) then it will remain zero and the spacetime will be FLRW (Ferrando *et al* 1992, corollary 1; but note that their statement that the anisotropic stress is invariant along u^a in general is misleading—from equations (31) and (40) of Ellis (1998) we have $\dot{\pi}_{\langle ab\rangle} \propto \theta\pi_{ab}$).

It is worth emphasizing that in applications of the above results to cosmology the motion of the fundamental observers must be identified with the congruence u^a. For example, in section 2.2 *all* Stephani models are conformally flat, and therefore conformally stationary, but for most of these spacetimes the fluid congruence is *not* aligned with the timelike CKV.

The matter content of the universe is not precisely known. Certainly, there are a large number of possible contributors, including hot and cold dark matter (in their various manifestations), electromagnetic fields, etc, as well as the more obvious radiation and baryonic matter. In particular, the type Ia supernova results of Perlmutter *et al* (1999) suggest that an important component may be a 'quintessential' scalar field. However, the forms of matter that are thought to contribute significantly to the energy–momentum tensor may be treated in general as perfect fluids. That is, their energy–momentum tensor may be written in the form

$$T_{ab} = \mu u_a u_b + p h_{ab}, \tag{3}$$

where μ and p are the energy density and pressure, u^a is the timelike velocity congruence of the fluid and h_{ab} is the spatial projection tensor associated with u^a. Scalar fields may also be written in this form, with u^a parallel to the gradient of the scalar field, provided that this is timelike (see section 3). Note that if several such components are present there is no reason why their fundamental congruences (the u^a's) should be parallel. If they are, then they may be treated as, effectively, a single perfect fluid with the energy densities and pressures added together. If not, the decomposition of the energy–momentum with respect to u^a for one such fluid (as in equation (14) of Ellis (1998)) will contain energy flux and anisotropic stress terms from the other fluids (again, see section 3). The fundamental observers will be associated with one such congruence. Usually the fundamental observers in standard models of the universe are associated with a dust-like ($p = 0$) component, with the result that the acceleration $\dot{u}^a$ of the fundamental congruence is zero. However, we wish to study the consequences of relaxing this assumption and consider models with acceleration. This acceleration must be caused by some non-gravitational force (typically pressure gradients for perfect fluid spacetimes, but in principle it could be the result of a coupling between the fluid and some other component such as the electromagnetic field).

With this in mind, in this paper we consider perhaps the two simplest generalizations of the dust hypothesis. First, we imagine that the dominant form of matter is a single irrotational perfect fluid. We do not specify what form of matter this corresponds to, but we allow pressure gradients that give rise to acceleration. Secondly, we consider cosmological models in which more than one matter component makes a significant contribution to the energy density and dynamics of the universe. Specifically we consider 'QCDM' models, containing a non-interacting mixture of radiation, dust (cold dark matter, CDM) and a scalar field (or a cosmological constant). The observers are associated with the CDM component, and are therefore geodesic. The difference between this and other theorems assuming geodesic observers is that the scalar field component can introduce effective energy flux and anisotropic stresses relative to the dust congruence, and so the matter need not behave as a perfect fluid.

In the following section we find all irrotational perfect fluid solutions admitting an isotropic radiation field for the fundamental observers, showing that they form a subclass of the Stephani spacetimes and are FLRW if and only if the acceleration vanishes. Then in section 3 we examine QCDM models and prove that such models must be homogeneous and isotropic if they admit an isotropic radiation field. Finally, in section 4 we emphasize the importance of acceleration for these results and show that the acceleration of the fundamental congruence is, in principle, detectable, and measurable, in galaxy surveys. Two appendices contain results relating to section 2.2.

2. The irrotational perfect fluid solutions

We wish to consider the constraints imposed by the existence of an isotropic radiation field for the fundamental observers on perfect fluid spacetimes in which the rotation of the fundamental congruence is zero. Since it follows from theorem 1 that the shear of the fundamental congruence must also be zero, we immediately know that all the acceptable solutions are members of the Stephani–Barnes family, which is the family of *all* shear-free, irrotational, expanding (or contracting) perfect fluids (see Barnes 1973, Krasiński 1989, 1997). It only remains, then, to impose condition (2) of theorem 1 and thus find the sub-class of Stephani–Barnes models which admit an isotropic radiation field for all fundamental observers.

The Stephani–Barnes family contains the Barnes solutions, which are of Petrov type D, and the Stephani models, which are conformally flat, although these two classes overlap where the Barnes solutions degenerate to type O (these solutions then become Stephani models with symmetry). The FLRW models are a subcase of these solutions. The Barnes spacetimes all possess symmetry (see below), whereas the Stephani spacetimes, in general, do not. In all cases the metric in comoving coordinates can be written in the form (see Krasiński (1989); although here we use the same notation as Krasiński (1997)):

$$\mathrm{d}s^2 = V^{-2}\left\{-(FV_{,t})^2\,\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right\} \tag{4}$$

where $F = F(t)$ and $V = V(t, x, y, z)$. $F(t)$ is arbitrary, but there are some restrictions on the form of V depending on the symmetries of the solution, and these will be discussed in due course (but the impatient reader may wish to note equations (10), (12), (13) and (15)). Expressions for the energy density and pressure can be found in Krasiński (1997).

The fluid velocity is given by (without loss of generality we can assume that $V > 0$)

$$u^a = \frac{V}{|FV_{,t}|}\delta^a{}_0, \tag{5}$$

with expansion

$$\theta = -\,\mathrm{sign}(V_{,t})\frac{3}{|F|}, \tag{6}$$

and acceleration

$$\dot{u}_0 = 0, \qquad \dot{u}_i = \frac{\partial}{\partial x^i}\ln\frac{V_{,t}}{V}, \tag{7}$$

where $i = 1, 2, 3$, $x^i = \{x, y, z\}$. Note that (6) differs from the expression usually given for the expansion (in Krasiński (1997), equations (4.1.4) and (4.9.6), for example) by the inclusion of the $-\,\mathrm{sign}(V_{,t})$ factor. Neglect of this factor is inconsistent since F enters the metric only quadratically, so the sign of the expansion cannot depend on the sign of F. The sign of θ *does* depend on that of $V_{,t}$, though: for the Friedmann subcase of the Stephani–Barnes models, for example, V is related to the scale factor $R(t)$ by $V = 1/R$, and $|FV_{,t}/V| = 1$ (see section 2.2), so that $\theta = 3\dot{R}/R = -3V_{,t}/V = -3\,\mathrm{sign}(V_{,t})/|F|$. This is important here because the constraint (2) contains the expansion. Note, too, that F is not a true degree of freedom parametrizing distinct spacetimes, but rather represents a coordinate freedom, corresponding to different choices of the time coordinate.

From (5)–(7), the condition (2) leads to the constraint

$$\frac{\partial^2}{\partial x^i \partial t}\ln V_{,t} = 0, \tag{8}$$

which is satisfied if and only if the function V has the form

$$V(t, x, y, z) = T(t)X(x, y, z) + Y(x, y, z), \tag{9}$$

where T, X and Y are arbitrary functions. This equation is the key additional constraint on the Stephani–Barnes solutions.

It is worth noting that it follows from (8) that the acceleration scalar is constant along the fluid flow for every observer, i.e. $\dot{u}_{,t} = 0$ (where $\dot{u}^2 = \dot{u}_a \dot{u}^a$), as can be seen by calculating $\dot{u}^2_{,t}$ from (7). In fact, it can be verified more generally that for any conformally stationary spacetime (i.e. a spacetime satisfying (1) and (2)), even with rotation, the acceleration scalar evolves according to

$$u^a \nabla_a \dot{u}^2 = \tfrac{2}{3} \dot{u}^b \widetilde{\nabla}_b \theta,$$

where $\widetilde{\nabla}$ denotes the spatially projected gradient (see Ellis 1998). It follows from equation (32) of Ellis (1998) that $\widetilde{\nabla}_b \theta = 0$ whenever the rotation vanishes (for a perfect fluid), which is the case for the Stephani–Barnes models.

Note that if V has the form (9) we can immediately write the metric in a manifestly conformally static form

$$\mathrm{d}s^2 = \frac{X^2}{V^2}\left\{-\mathrm{d}\tau^2 + X^{-2}(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)\right\},$$

where $\mathrm{d}\tau = T_{,t} F\, \mathrm{d}t$, which shows that these models will indeed be conformally stationary, as required by theorem 1. We now discuss each subcase in turn.

2.1. The Barnes solutions

The Barnes solutions (Barnes 1973) all have spherical, plane or hyperbolic symmetry (i.e. they possess three-dimensional isometry groups acting on two-dimensional orbits, cf section 2.2). The restrictions on the metric function V depend on which of these symmetries the spacetime possesses (see Krasiński 1997). For the solutions with spherical symmetry (the Kustaanheimo–Qvist solutions) or planar symmetry we introduce a new independent variable $u(x, y, z)$ defined by $u = r^2 = x^2 + y^2 + z^2$ in the spherical case and $u = z$ in the planar case. Then V is defined by $V = V(t, u)$, subject to the condition

$$\frac{\partial^2 V}{\partial u^2} = f(u)V^2, \tag{10}$$

where f is an arbitrary function. Since $V = V(t, u)$ we know from (9) that in order to admit an isotropic radiation field for all observers V must have the form

$$V(t, u) = T(t)X(u) + Y(u). \tag{11}$$

Equation (10) then imposes a constraint on the functions X and Y, which will be outlined below.

For the solutions with hyperbolic symmetry the constraint on V is very similar. This time we introduce the variable $u = x/y$. V can then be written

$$V(t, x, y) = yW(t, u), \tag{12}$$

where W satisfies

$$\frac{\partial^2 W}{\partial u^2} = f(u)W^2, \tag{13}$$

with f once again a free function. The condition (9) now gives

$$V(t, x, y) = T(t)X(x, y) + Y(x, y) = yW(t, u).$$

Dividing by y and redefining X and Y in an obvious fashion we obtain

$$W(t, u) = T(t)\tilde{X}(u) + \tilde{Y}(u), \tag{14}$$

in which $\tilde{X}$ and $\tilde{Y}$ are again constrained by (13).

For all three symmetries of the Barnes solutions, then, the constraints on the metric function V essentially reduce to the second-order differential equations (10) or (13). Imposing the condition (9) introduces the additional constraint (11) or (14). Substitution of (11) or (14) into (10) or (13), respectively, and differentiating twice with respect to time, dividing by $T_{,t}$ each time (and recognizing that $T_{,t} \neq 0$, $X \neq 0$, so that $V_{,t} \neq 0$ in (4)), leads directly to the condition

$$f(u) = 0.$$

Barnes solutions with $f(u) = 0$ are conformally flat (Krasiński 1997, p 142) and are therefore actually a subcase of the Stephani models. That is, proper Barnes spacetimes can be ruled out: they do not admit an isotropic radiation field. It only remains to apply the condition (9) to the Stephani models, which we do in the next section.

2.2. The conformally flat solutions

The conformally flat sub-case is the entire class of conformally flat, expanding, perfect fluid solutions, and is the Stephani solution (Stephani 1967a, b). The function V is most often written in the form (see Krasiński 1997, Barrett and Clarkson 1999a, b):

$$V(t, x, y, z) = \frac{1}{R(t)}\left(1 + \tfrac{1}{4}k(t)|\boldsymbol{x} - \boldsymbol{x}_0(t)|^2\right). \tag{15}$$

In general, the five functions of time in V are free. For our purposes, however, it turns out to be more convenient to use

$$V = a(t) + b(t)r^2 - 2\boldsymbol{c}(t)\cdot\boldsymbol{r}, \tag{16}$$

as in Barnes (1998), again with five free functions (we adopt three-dimensional vector notation, so that $\boldsymbol{c} = (c_1, c_2, c_3)$, for example). In fact, this form is slightly more general than (15)—see Barnes (1998).

We must be able to write V in the form (9) for the spacetime to admit an isotropic radiation field for all fundamental observers. From (15) or (16) it is clear that the functions $X(x^i)$, and $Y(x^i)$ in (9) can be at most quadratic in x^i. Writing X and Y as quadratics in (9), and equating in (16) all powers of x^i, we obtain the following constraint equations:

$$\begin{aligned} a(t) &= a_1T(t) + a_2, \\ b(t) &= b_1T(t) + b_2, \\ \boldsymbol{c}(t) &= \boldsymbol{c}_1T(t) + \boldsymbol{c}_2, \end{aligned} \tag{17}$$

where $T(t)$ is a free function of time and the $a_{1,2}$, $b_{1,2}$ and $\boldsymbol{c}_{1,2}$ are ten independent constants. Not all of a_1, b_1 and $\boldsymbol{c}_1$ can be zero (in order that $V_{,t} \neq 0$ in (4)).

Equations (17), along with (4) and (16), provide the complete set of irrotational perfect fluid spacetimes admitting an isotropic radiation field. Not all of the possible choices of parameters give rise to distinct spacetimes, though, and we outline in appendix A how coordinate transformations may be used to eliminate many of the parameters in (17), and determine when the models can be reduced to a manifestly spherically symmetric form ($\boldsymbol{c} = \mathbf{0}$). Given the forms for a, b and $\boldsymbol{c}$ in (17) we can equate (15) with (16) to find the corresponding constraints on R, k and $\boldsymbol{x}_0$ in (15). This we do in appendix B.

At this point we can say that perfect fluid spacetimes admitting an isotropic radiation field for all fundamental observers are FLRW if and only if the acceleration of the fundamental observers is zero. This follows because the Stephani models with zero acceleration are FLRW (see Krasiński 1997, although it can easily be seen from (7): if $\dot{u} = 0$ then $V = T(t)X(x^i)$ for some functions T and X, showing that a, b and $\boldsymbol{c}$ depend on the single free function T and so must be FLRW by the results of the next section). Thus we have proved:

Theorem 3. *The irrotational perfect fluid spacetimes admitting an isotropic radiation field for the fundamental observers are Stephani models with the free functions restricted by (17) (or (A3)). These spacetimes are FLRW if and only if the acceleration of the fundamental observers is zero.*

It is worth noting that all of the models admitting an isotropic radiation field are manifestly conformal to (part of) the Einstein static spacetime (cf Barrett and Clarkson 1999a) once they have been transformed so that $\boldsymbol{c} = c\hat{\boldsymbol{z}}$ as outlined in appendix A. From (16) and (A3) we obtain

$$V_{,t} = a_1 T_{,t}\left(1 + \frac{b_1}{a_1} r^2\right),$$

since $\boldsymbol{c}_{,t} = \mathbf{0}$ when $\boldsymbol{c} = c\hat{\boldsymbol{z}}$ (as noted in appendix A we can assume $a_1 \neq 0$). Changing the time coordinate via $\mathrm{d}t \mapsto a_1 T_{,t} F\,\mathrm{d}t$, the metric (4) becomes

$$\mathrm{d}s^2 = \frac{(1 + \frac{1}{4}\Delta r^2)^2}{V(z, r, t)^2}\left\{-\mathrm{d}t^2 + \frac{1}{(1 + \frac{1}{4}\Delta r^2)^2}(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)\right\},$$

where $\Delta = 4b_1/a_1$. The factor in braces is the Einstein static metric. Barrett and Clarkson (1999a) used this conformal relationship to simplify the study of the observational characteristics of a subset of these models.

2.3. Symmetry and thermodynamic schemes

Having obtained the conditions (17) for a Stephani model to admit an isotropic radiation field, it is possible to say immediately that all such spacetimes possess (at least) a three-dimensional symmetry group acting on two-dimensional orbits (just as for the general Barnes models). This follows from the work of Barnes (1998), who showed that the dimension of the isometry group of any Stephani spacetime is determined by the dimension d of the linear space spanned by the five free functions a, b and $\boldsymbol{c}$:

(a) if $d = 4$ or 5 (i.e. at least four of the free functions are linearly independent), then the spacetime has no Killing vectors;
(b) if $d = 3$ there is a one-dimensional isometry group;
(c) if $d = 2$ there is a three-dimensional isometry group acting on two-dimensional orbits;
(d) if $d = 1$ there are six Killing vectors and the spacetime is Robertson–Walker.

It is clear from (17) that a, b and $\boldsymbol{c}$ depend on (at most) only two functions of time: $f_1(t) = T(t)$ and the constant function $f_2(t) \equiv 1$ (since $V_{,t} \neq 0$ we must have $T_{,t} \neq 0$, so that these are necessarily linearly independent). Thus, $d = 2$ and the solutions have three-dimensional isometry groups as claimed. If $a_2 = b_2 = \boldsymbol{c}_2 = 0$ in (17) then $d = 1$ and the spacetime is FLRW (see appendix A).

It follows further from this and the work of Bona and Coll (1988) (see also Krasiński *et al* 1997) that all Stephani models that admit an isotropic radiation field for all fundamental observers also admit a strict thermodynamic scheme (that is, entropy and temperature functions can be found that depend on the energy density and pressure and satisfy the first law of thermodynamics as embodied by the Gibbs equation). The converse is not true, however, since there are Stephani spacetimes with $d = 2$ (which must admit a thermodynamic scheme) that cannot have an isotropic radiation field (these are models for which the second independent function $f_2(t)$ is not restricted to be 1). So, we have the following corollary to theorem 3:

Corollary 1. *An irrotational perfect fluid spacetime that admits an isotropic radiation field has spherical, planar or hyperbolic symmetry and admits a strict thermodynamic scheme.*

On the subject of thermodynamics, let us mention for completeness that the thermodynamic scheme occurring most often in the literature is that of a barotropic equation of state. It is known that the only Stephani models with a barotropic EOS are precisely the FLRW models (Bona and Coll 1988, Krasiński 1997). Thus, the only spacetimes with a barotropic EOS admitting an isotropic radiation field are FLRW models. This also follows (when $\mu + p \neq 0$) from a theorem of Coley (1991).

3. QCDM models

The type Ia supernova results of Riess *et al* (1998), Schmidt *et al* (1998) and Perlmutter *et al* (1999), which suggest that the expansion of the universe is accelerating, have led to an increased interest in cosmological models in which a significant contribution to the energy density comes from either a cosmological constant or a scalar field (quintessence component), which is capable of driving the expansion (Peebles and Ratra 1988, Ratra and Peebles 1988, Caldwell *et al* 1998, Zlatev *et al* 1999). In QCDM models the matter is an admixture of non-interacting cold dark matter, i.e. dust, and a scalar field. The fundamental observers (galaxies) are implicitly identified with the geodesic congruence of the CDM. Note that there is no reason *a priori* why the scalar field gradient (which defines a natural 'velocity' field) should be aligned with the CDM congruence (although it will turn out that they are aligned if the fundamental observers see an isotropic radiation field).

It is interesting to ask whether the EGS theorem can be extended to this case. We demonstrate that it can by proving the following theorem:

Theorem 4. *Any solution to Einstein's equations in which the matter consists of non-interacting radiation, expanding dust (CDM), and a scalar field (or cosmological constant), and for which the dust sees an isotropic radiation field, must either be an FLRW model, or have the gradient of the scalar field orthogonal to the dust congruence.*

(Note that the latter possibility means that the gradient of the scalar field is spacelike, and is usually rejected as unphysical—although see below.)

Proof. We may divide this proof into two parts: first we demonstrate from Einstein's equations in the 1 + 3 formalism that any energy flux component with respect to the CDM frame must be zero if the CDM observers see isotropic radiation, then we show that the contribution to the energy flux (with respect to the CDM frame) from the scalar field is zero if and only if the gradient of the scalar field is parallel (or orthogonal) to the CDM velocity u^a_{CDM}, so we deduce that the velocity fields are parallel (or orthogonal). The case where the field gradient is orthogonal to u^a_{CDM} is probably unphysical, and will be rejected. Thus, the mixture of radiation, CDM and scalar field can be written as a single perfect fluid with geodesic fundamental congruence $u^a = u^a_{CDM}$, and it follows from the results of section 2 that the model is necessarily FLRW. Throughout this section equation numbers of the form (E32) will refer to the lectures of Ellis (1998).

Since the radiation is isotropic for the dust observers u^a the energy–momentum tensor for radiation may be written in the perfect fluid form (3) with $p = \frac{1}{3}\mu$, and the total energy–momentum tensor is

$$T_{ab} = \underbrace{\mu u_a u_b + \tfrac{1}{3}\mu h_{ab}}_{\text{radiation}} + \underbrace{\rho u_a u_b}_{\text{CDM}} + \underbrace{\phi_{,a}\phi_{,b} - g_{ab}\left(\tfrac{1}{2}\phi_{,c}\phi^{,c} + \Phi(\phi)\right)}_{\text{scalar field}}, \tag{18}$$

where $\Phi(\phi)$ is the scalar field potential (often assumed to be zero, in which case the scalar field can be interpreted as a stiff perfect fluid). Note that the cosmological constant case can be included by setting $\phi = \Lambda =$ constant, $\Phi(\phi) = \phi$.

(a) The fundamental congruence u^a is geodesic ($\dot{u}^a = 0$) because the CDM component does not interact with the other matter. This, in fact, implies that the rotation of u^a must also vanish: from the momentum conservation equation for the radiation, we can write

$$\dot{u}_a = -\tfrac{1}{4}\widetilde{\nabla}_a \ln \mu = 0$$

(where $\widetilde{\nabla}_a$ denotes the spatially projected gradient), so that (using (E27))

$$0 = \widetilde{\nabla}_{[a}\dot{u}_{b]} = \frac{1}{4}\widetilde{\nabla}_{[b}\widetilde{\nabla}_{a]} \ln \mu = \frac{1}{4}\omega_{ab}\frac{\dot{\mu}}{\mu} = \frac{1}{3}\omega_{ba}\theta,$$

and we see that $\omega_{ab} = 0$ when $\theta \neq 0$.
When $\dot{u}_a$ and ω_{ab} are zero, equation (2) becomes

$$\nabla_{[a}(\theta u_{b]}) = u_{[b}\nabla_{a]}\theta = 0.$$

This implies (since $\nabla_a\theta = \widetilde{\nabla}_a\theta - \dot{\theta}u_a$) that

$$\widetilde{\nabla}_a\theta = 0, \tag{19}$$

(i.e. the expansion is homogeneous). From the constraint equation relating the divergence of the shear to other kinematical quantities (E32) we see that any energy flux component with respect to the CDM velocity field must vanish:

$$q_a = \tfrac{2}{3}\widetilde{\nabla}_a\theta = 0, \tag{20}$$

This is the key step in the proof.

(b) Decomposing (18) with respect to u^a we find that the relative energy flux component is

$$0 = q_a = -h_a{}^b u^c T_{bc} = -\dot{\phi}\widetilde{\nabla}_a\phi. \tag{21}$$

So $q_a = 0$ if $\dot{\phi} = 0$ (the scalar field gradient is orthogonal to u^a, and therefore spacelike), or if $\widetilde{\nabla}_a\phi = 0$ (the scalar field gradient is parallel to u^a). We take the latter case to be most important since the gradient of a scalar field is usually assumed to be timelike.
Since $\nabla_a\phi = \widetilde{\nabla}_a\phi - u_a\dot{\phi} = -u_a\dot{\phi}$ it is possible to write (18) in the form of a single perfect fluid (3) with a geodesic, shear-free, rotation-free velocity field; it is thus an FLRW model by section 2. □

It is easy to see from the above proof that the fact that the fundamental observers correspond to dust-like matter was not used, only that they followed geodesics. Hence the above result applies for more general perfect fluids in place of the CDM component, as long as the fundamental congruence is geodesic.

The idea of a spacelike scalar field gradient seems physically unappealing. However, such a field can (depending on the potential Φ) satisfy the weak, strong and dominant energy conditions. The strong energy condition will be satisfied if and only if $\Phi(\phi) \leqslant 0$ everywhere, whereas as the weak and dominant energy conditions will be satisfied if $\Phi(\phi) \geqslant 0$, although not only so. Thus, a massless scalar field (stiff perfect fluid) with spacelike gradient satisfies all energy conditions. It should be borne in mind, though, that scalar fields arising in cosmological contexts often fail to satisfy the energy conditions. This case may deserve further consideration. As can easily be seen, the scalar field component gives rise to anisotropic stresses in the energy–momentum tensor, so such spacetimes are not FLRW.

4. Conclusions

We have proved that the irrotational perfect fluid spacetimes admitting an isotropic radiation field are Stephani models restricted by (17) (see also equations (A3)), and are FLRW if and only if the acceleration $\dot{u}^a$ of the fundamental congruence is zero (theorem 3). It follows from the fact that the constraints (17) depend on only two independent functions of time that all of the acceptable models possess three-dimensional symmetry groups acting on two-dimensional orbits (i.e. have spherical, planar or hyperbolic symmetry) and therefore possess a thermodynamic interpretation. We have also shown that spacetimes containing a mixture of radiation, dust and scalar field (QCDM models), for which the dust observers see the radiation as isotropic, must always be homogeneous and isotropic (theorem 4) unless the scalar field gradient is spacelike and orthogonal to the CDM congruence—a possibility we reject as unphysical. This result also relies on the geodesic nature of the fundamental congruence.

Crucial, therefore, to the proof of homogeneity and the verification of the cosmological principle is the non-acceleration of the fundamental observers. Despite the intuitive appeal of cosmological models in which the fundamental observers are associated with a dust-like matter component, it is unacceptable to simply assume that we are geodesic observers, especially when such an assumption is, in principle, testable. Acceleration leaves a characteristic dipole signature in the redshifts of nearby galaxies that may be detectable using galaxy surveys. The physical principle underlying this effect is easy to see. For a set of uniformly accelerated observers (‘galaxies’) in Minkowski space each observer will see other galaxies redshifted or blueshifted in a dipole pattern, with the blueshifted galaxies lying in the direction of the acceleration, because during the light-travel time between galaxy and observer the observer’s velocity has increased relative to the velocity at emission, so that the galaxies the observer is travelling towards are blueshifted, and those it is travelling away from redshifted. It also follows from this that the magnitude of the dipole increases with distance, simply because the light-travel time from more distant galaxies is larger. In a cosmological context this acceleration dipole must be added to other terms contributing to the redshift of nearby objects, in particular the expansion. The method of Kristian and Sachs (1966) and MacCallum and Ellis (1970) gives, for any cosmological model with fundamental congruence u^a, the lowest-order term in the redshift z as a function of distance r and direction e^a (a spacelike unit vector orthogonal to u^a, denoting the direction of observation)

$$z = H_0 r \left(1 - \frac{\dot{u}_a e^a}{H_0} + \frac{\sigma_{ab} e^a e^b}{H_0} \right) \Bigg|_0 + \mathcal{O}(r^2), \tag{22}$$

where $H_0 = \frac{1}{3}\theta_0$ is Hubble's constant and the last term in parentheses indicates the quadrupole introduced by the presence of shear. In (22) r can be any cosmological distance measure (area distance, for example) because for small r all such measures agree to first order. Note that just as the monopole (expansion) term increases linearly with distance according to the Hubble law, so does the acceleration dipole. This is important, because it allows the acceleration dipole to be distinguished from any dipole resulting from the peculiar velocity of our galaxy with respect to the cosmological average rest frame (usually identified with the CMB frame). Equation (22) applies in this rest frame, and any peculiar motion results in a Doppler shift for each galaxy, which introduces an additional dipole component into the galaxy redshifts. This dipole is just a constant depending only on the peculiar velocity of our galaxy. A boost to the 'correct' rest frame can eliminate this constant component, but cannot remove the acceleration dipole because it is distance dependent. It is important to note in this context that the acceleration referred to here is not the same as the 'acceleration dipole' resulting from the gravitational attraction by the Great Attractor overdensity, which is often calculated using galaxy surveys (see Schmoldt *et al* 1999).

Galaxy surveys are often used to measure a possible bulk flow of the local universe, that is, the difference, if any, between the rest frame of the local universe and the CMB frame, which in standard cosmological models should be the same (see Willick 1998). A simple extension of these techniques (Clarkson *et al*, in preparation) permits the acceleration to be constrained by observations. However, preliminary results suggest that the constraints on $\dot{u}^a$ are quite weak: it appears not to be possible to conclude definitively that we are geodesic observers. The accuracy of $\dot{u}^a$ determinations is limited both by uncertainties in the distance estimates to galaxies as well as the peculiar velocities of galaxies.

Even if the acceleration was measured to be zero, it is still necessary to show that there are no anisotropic stresses (Ferrando *et al* 1992) before the cosmological principle can be verified. It follows from equation (31) of Ellis (1998) that this is equivalent to determining that the electric part of the Weyl tensor is zero. It is not clear how this may be achieved using observations.

Of course, the Copernican principle (which is a vital element of EGS-type theorems, allowing the high isotropy of the CMB here to be assumed for other points in the universe) remains a purely philosophical assumption. It has been suggested by a number of authors (see Goodman 1995, for example) that the Sunyaev–Zeldovich effect might be used to place constraints on the anisotropy of the CMB at distant positions, but it is not obvious that such observations will provide a definitive verification of the Copernican principle. Nevertheless, the arguments in favour of the Copernican principle are quite powerful, and it is a much weaker assumption than the cosmological principle. Note that if the acceleration *is* measured to be zero here, the Copernican principle must also be applied to give geodesic observers everywhere for the results of this paper (and the other EGS papers) to hold.

Finally, one might expect that the 'almost' version of theorem 3 would lead to spacetimes that are almost the Stephani models of (17). However, when the assumption of geodesic observers is relaxed it is no longer possible to constrain the rotation to be small, and the class of perfect fluid spacetimes with an almost isotropic CMB may include examples with distinctly non-zero rotation, unless other constraints are brought to bear. Actually, it is possible that *all* spacetimes admitting an isotropic radiation field are irrotational (see Coley 1991, section 2.2), although this has not been shown definitively. It would be interesting to determine the class of all perfect fluid spacetimes admitting an isotropic radiation field, and if it turns out that they are indeed all irrotational then the Stephani spacetimes defined by (17), (4) and (16) are indeed the complete set.

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Appendix A. Coordinate transformations and the Stephani spacetimes

On the face of it the Stephani models admitting an isotropic radiation field in (17) depend on one free function and ten free parameters. However, it is possible to use coordinate transformations on the spacetime to eliminate many of these parameters, resulting in a considerable simplification. As is shown in Barnes (1998), conformal transformations of the coordinates on the hypersurfaces of constant time preserve the form of the metric but change the free functions a, b and $\boldsymbol{c}$. These transformations can be thought of as acting on the five-dimensional space spanned by a, b and $\boldsymbol{c}$ and constitute the Lorentz group in five dimensions, $SO(4, 1)$: they leave $-ab+|\boldsymbol{c}|^2$ invariant (a and b are 'null coordinates'). It will be convenient here to let $a = \alpha + \beta$ and $b = \alpha - \beta$ (so that the transformations preserve $-\alpha^2 + \beta^2 + |\boldsymbol{c}|^2$), and to adopt 5-vector notation: $q^\mu = (\alpha, \beta, \boldsymbol{c})$. We will use the terms 'rotation' and 'boost' to refer to the transformations on q, and will call q timelike, spacelike or null if $-\alpha^2+\beta^2+|\boldsymbol{c}|^2$ is negative, positive or zero, as usual. Then it is easy to visualize the transformations on the free functions by imagining the 'mass hyperboloids' of representations of the Lorentz group in the usual way: a timelike vector can always be boosted so that it has the form $(\alpha, 0, \mathbf{0})$, whereas a spacelike vector can be boosted and rotated into $(0, \beta, \mathbf{0})$, for example.

In addition to the Lorentz transformations we also have the freedom to change basis in the function space spanned by the free functions. For the spacetimes of interest here, described by (17), it is desirable to preserve $f_2(t) = 1$, so that the basis change is

$$T \mapsto \gamma T + \delta. \tag{A1}$$

In 5-vector notation equations (17) become

$$q^\mu(t) \equiv (\alpha, \beta, \boldsymbol{c}) = q_1^\mu T(t) + q_2^\mu \tag{A2}$$

(with q_1 and q_2 constant vectors), and the goal is to reduce as many of the components of q_1^μ and q_2^μ to zero as possible using Lorentz transformations in the five-dimensional space containing q_1 and q_2, and the basis change (A1). Note that the FLRW ($d = 1$) subcase of (17) is characterized by the linear dependence of q_1 and q_2.

It is easy to see that $\boldsymbol{c}$ (which breaks the spherically symmetry of the metric (4)) may always be reduced to the form $\boldsymbol{c} = c\hat{\boldsymbol{z}}$ ($\hat{\boldsymbol{z}} = (0, 0, 1)$), with c a constant: perform a spatial 4-rotation to reduce q_1^μ to $q_1^\mu = (\alpha_1, \beta_1, \mathbf{0})$, then a spatial rotation amongst the $\boldsymbol{c}$-components (which obviously leaves q_1 unaffected) to give $q_2^\mu = (\alpha_2, \beta_2, c\hat{\boldsymbol{z}})$.

It is possible in general to make further simplifications, but precisely how q_1 and q_2 are simplified depends on whether they are spacelike, timelike or null. For example, if either q_1 or q_2 is timelike (or may be made timelike by a transformation (A1)) it is possible to reduce the model to manifestly spherically symmetric form ($c = 0$): boost so that the timelike vector, say q_1, becomes $q_1 = (\alpha_1, 0, \mathbf{0})$ and rotate spatially so that the $\boldsymbol{c}$-components of the other vector are also zero, $q_2 = (\alpha_2, \beta_2, \mathbf{0})$ (we could then use (A1) to eliminate more of these constants).

To summarize, we have demonstrated that it is always possible to reduce the Stephani models of (17) to the form

$$a(t) = a_1 T(t) + a_2,$$
$$b(t) = b_1 T(t) + b_2, \tag{A3}$$
$$\boldsymbol{c}(t) = c\hat{\boldsymbol{z}},$$

(where we have transformed back from α and β to a and b), and when either of the q_1 or q_2 is (or may be made) timelike we can set $c = 0$.

Finally, note that we may always assume that $a_1 \neq 0$ in (A3), because if $a_1 = 0$ then $b_1 \neq 0$, otherwise $V_{,t} = 0$, and it is possible to perform a coordinate inversion $\boldsymbol{x} \mapsto \boldsymbol{x}/r^2$ that interchanges a and b. This does not exhaust the possibilities for simplification: we could, for example, use (A1) to set $a_2 = 0$.

Appendix B. The constraints on R, k and $\boldsymbol{x}_0$

To find the constraints on R, k and $\boldsymbol{x}_0$ in (15) corresponding to (17) first equate powers of x^i in (15) and (16) to obtain

$$a(t) = \frac{1}{R} + \frac{k}{4R}|\boldsymbol{x}_0|^2, \tag{B1}$$

$$b(t) = \frac{k}{4R}, \tag{B2}$$

$$\boldsymbol{c}(t) = \frac{k}{4R}\boldsymbol{x}_0. \tag{B3}$$

Solving these equations for R, k and $\boldsymbol{x}_0$ gives

$$R(t) = \frac{b}{ab - |\boldsymbol{c}|^2}, \tag{B4}$$

$$\frac{1}{4}k(t) = \frac{b^2}{ab - |\boldsymbol{c}|^2}, \tag{B5}$$

$$\boldsymbol{x}_0 = \frac{\boldsymbol{c}}{b}, \tag{B6}$$

which are valid whenever $ab - |\boldsymbol{c}|^2 \neq 0$ (otherwise V cannot be written in the form (15)).

To impose the constraints (17) perform the transformations of appendix A so that $\boldsymbol{c} = c\hat{\boldsymbol{z}}$ as in (A3)) and $a_1 \neq 0$. Then $|\boldsymbol{c}|^2 = c^2$, and b and a are related by

$$b = \frac{b_1}{a_1}a + \left(b_2 - \frac{b_1 a_2}{a_1}\right) \equiv \gamma a + \delta, \tag{B7}$$

where $\gamma = b_1/a_1$ and $\delta = b_2 - b_1 a_2/a_1$ are constants. Using this in (B4) and (B5) leads, after some rearrangement, to a quadratic relationship between k and R:

$$\left(\tfrac{1}{4}k\right)^2 - (\gamma + \delta R)\left(\tfrac{1}{4}k\right) - \gamma c^2 R^2 = 0. \tag{B8}$$

In addition to this constraint relating k and R we can trivially rewrite (B3) or (B6) as

$$\boldsymbol{x}_0 = \frac{4R}{k}c\hat{\boldsymbol{z}}. \tag{B9}$$

Equations (B8) and (B9) are the constraint equations on R, k and x_0 corresponding to equations (17), or rather (A3). If desired (B8) can be solved to obtain

$$\tfrac{1}{4}k = \tfrac{1}{2}\left[\gamma + \delta R \pm \sqrt{(\gamma + \delta R)^2 + 4\gamma c^2 R^2}\right].$$

From (B8) it is clear that the spherically symmetric Stephani spacetimes admitting an isotropic radiation field, for which $c = 0$, satisfy

$$\tfrac{1}{4}k\left(\tfrac{1}{4}k - (\gamma + \delta R)\right) = 0,$$

which has the solutions $k = 0$ and $R(t)$ free (flat Friedmann), or k linearly related to R,

$$\tfrac{1}{4}k(t) = \gamma + \delta R(t),$$

with $R(t)$ again free (when $\delta = 0$ these become Friedmann models with curvature $k = 4\gamma$). The Stephani models studied by Barrett and Clarkson (1999a, b) are members of this class (with R depending quadratically on t), which explains the isotropy of the microwave background found for those spacetimes.

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