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Conditions for equivalence between the Hu–Washizu and related formulations, and computational behavior in the incompressible limit

J.K. Djoko [a], B.P. Lamichhane [b], B.D. Reddy [c,*], B.I. Wohlmuth [b]

[a] Department of Mathematics and Applied Mathematics, University of Cape Town, 7701 Rondebosch, South Africa
[b] Institute of Applied Analysis and Numerical Simulations (IANS), Universität Stuttgart, Pfaffenwaldring 57, 70529 Stuttgart, Germany
[c] Faculty of Science, University of Cape Town, 7701 Rondebosch, South Africa

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Abstract

The relationship of the Hu–Washizu mixed formulation to other mixed and enhanced formulations is examined in detail, in the context of linear elasticity, with a view to presenting a unified framework for such formulations. The Hu–Washizu formulation is considered in both its classical form and in a modified form that is suited to establishing well-posedness in the incompressible limit. Recently established theoretical results on uniform well-posedness are studied computationally by considering a range of numerical examples, which illustrate where appropriate the good behavior predicted by the theory, as well as the locking behavior that is evident in those problems in which the conditions for stability and convergence are not satisfied.

Keywords: Hu–Washizu; Mixed finite elements; Enhanced strains; Low-order approximations; Equivalence; Numerical simulations

1. Introduction

The use of low-order elements in finite element computations remains a popular feature in finite element analyses of problems in solid mechanics. Four-noded quadrilateral elements and eight-noded hexahedra in two and three dimensions, respectively, have a key drawback in that they lead to locking in the incompressible limit; in other words, they do not possess the property of being uniformly convergent. In addition, even in compressible problems the use of these standard elements leads to poor accuracy particularly in bending-dominated problems, when coarse meshes are used.

Various procedures have been proposed for circumventing these difficulties. Among these are stabilization methods such as those, for example, that are combined with underintegration [4,10], and the classes of Petrov–Galerkin methods in which stabilization is achieved through a relaxation of the requirement that the spaces of test and trial functions be identical, and Galerkin least squares methods in which the introduction of additional terms serves to stabilize the problem while preserving consistency [14–17]. Brezzi and Fortin [8] have undertaken a detailed abstract analysis that is applicable to a wide range of such approaches.

Methods associated with the enrichment or enhancement of the strain or stress field by the addition of carefully chosen basis functions have proved to be highly effective and popular. The key work dealing with enhanced assumed strain formulations is that of Simo and Rifai [35]. This method, which may be regarded as a generalization of the method of

* Corresponding author. Tel.: +27 21 650 2711; fax: +27 21 650 2710.
E-mail address: bdr@science.uct.ac.za (B.D. Reddy).

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incompatible modes [39,37], has been extended to nonlinear problems in two and three dimensions by Simo et al. [30,31]. Extensive computational studies indicate that the method provides a promising approach for overcoming the difficulties referred to earlier, though computational investigations have shown (see, for example, [2,28]) that the development of elements and bases that are robust for a wide range of problems involving near-incompressibility or incompressibility is not a trivial matter.

The assumed stress approach [26] leads to a formulation very similar to that based on enhanced strains, and in fact the two are equivalent under certain conditions [1,6]. Even the underintegrated elements with additional stabilization can be derived from enhanced strain elements for certain element shapes [19], so that a clear line cannot always be drawn between these methods. This holds particularly for affine-equivalent meshes.

Reddy and Simo [27] have carried out a detailed analysis of the convergence of enhanced assumed strain methods, for affine-equivalent meshes, and for the compressible and incompressible cases. They established an a priori error estimate for displacements that confirms convergence at the standard linear rate. Braess [5] has re-examined the sufficient conditions for convergence, in particular relating the stability condition to a strengthened Cauchy inequality, and elucidating the influence of the Lamé constant λ.

Arunakirinathar and Reddy [3] have extended the results in [27] to include consideration of isoparametric elements. The case of limiting compressibility was not studied by these authors, but has been the subject of a recent analysis by Braess et al. [6], in which λ-independent asymptotic convergence of the displacement error is obtained, for a class of meshes. These authors also establish the conditions under which the method of enhanced assumed strains is equivalent to the Hellinger–Reissner formulation, and derive a residual-based a posteriori error estimate.

Analysis of the near-incompressible problem for neo-Hookean materials, with finite element error estimates, has been recently presented in [7].

The purpose of this work is to revisit three-field mixed formulations as an approach to overcoming the problems referred to above. In the context of elasticity the starting point is the Hu–Washizu formulation [18,38], in which the unknown variables are displacement, strain, and stress. This formulation, which was in fact first introduced by Fraeijs de Veubeke [11] (see [12,13], a reprint of [12], and the historical comments in [9]), serves as the point of departure for the development of enhanced strain formulations, whether the original method of Simo and Rifai [35], or further extensions such as that due to Kasper and Taylor [21,22]. In these latter works the method of mixed enhanced strains is presented, for linear and nonlinear problems, respectively. The difference between this and the original method of enhanced strains lies in the inclusion of the stress as an unknown variable, together with displacement and enhanced strain (in contrast, the method of Simo and Rifai leads to elimination of the stress, which must be recovered a posteriori). Numerical results are shown to be of comparable quality to those obtained using enhanced assumed strains, with the advantage that the stress is obtained directly as part of the solution.

Also of relevance, as a method having the Hu–Washizu formulation as its basis, is the strain gap method due to Romano et al. [29]. While these authors have carried out an analysis of the Hu–Washizu formulation, their analysis is valid only for compressible materials.

The numerical evidence in favor of three-field formulations is compelling, but there remains the question of the precise conditions under which these formulations are uniformly convergent, in the incompressible limit. This has been the subject of a recent study [23]. In this work, the mathematical basis for establishing well-posedness of the Hu–Washizu formulation and its derivatives is investigated in detail. It is shown there that in order to ensure uniform convergence for bases that are in common use, it is necessary to approach the problem through a modified formulation that is equivalent to that obtained from the standard Hu–Washizu functional in the continuous case, but which may differ from it in the discrete setting. The modified formulation leads to the construction of a family of modified Hu–Washizu formulations, parametrized by a scalar α, with the classical problem recovered as the special case $\alpha = 1$. Well-posedness is established for the family of alternative formulations in circumstances in which this may not be possible for the classical Hu–Washizu formulation.

A crucial element of the analysis of the alternative formulation in [23] is the need to establish control over the spherical part of the stress, which is the Lagrange multiplier variable in the Hu–Washizu formulation. In the discrete setting the spherical component is not necessarily a member of the space of discrete stresses, so it is necessary to introduce the notion of the discrete spherical and deviatoric parts of the stress space; these two components are by definition subspaces of the space of stresses.

A further significant feature of the analysis in [23] is the requirement that the discrete space of displacements V_h forms, with the trace of $\mathrm{sph}_h S_h$, the discrete spherical part of S_h, a Stokes-stable pair, in the sense that these two spaces satisfy the discrete inf–sup condition with a constant independent of h and λ, the Lamé parameter. When V_h is the space of piecewise bilinear or Q_1 functions, then $\mathrm{tr}\,\mathrm{sph}_h S_h$ is the space of piecewise constants, so that checkerboard modes are present. This will be illustrated numerically in the present work. As with the Stokes problem, in which the pressure space of piecewise constants has to be modified to extract from it the checkerboard modes, it can be shown that the lack of stability resulting from the presence of a checkerboard mode is confined to the stress, and does not affect the displacement and strain approximations. This result is important in the computational context, especially in situations in which the displacement is the primary variable of interest.

The focus in [23] is the establishment of conditions under which the relevant formulations possess unique solutions, and under which finite element approximations are convergent. There remain, though, various questions that require further investigation, and these are the subject of the present work.

First, we show how a modified Hu–Washizu formulation may be derived from the original saddlepoint formulation by a partial imposition of the volumetric part of the elasticity equation. It is shown further that this corresponds to the parametrized formulation with $\alpha = 0$. Interpolation between this and the classical formulation (that is, the case $\alpha = 1$) then leads to the parametrized formulation that has been introduced in [23] without similar motivation.

The second key contribution of this work is an investigation of the relationships between various discrete mixed and enhanced formulations, and the conditions under which they are equivalent. Some equivalence results are well-known while others are presented for the first time. These are set out in such a way, diagrammatically, that the relationship between any two formulations contained in the diagram may be deduced immediately. Furthermore, the set of equivalence relationships together with information about the well-posedness of a particular formulation permits conditions for well-posedness of an equivalent formulation to be deduced.

The third contribution entails a detailed numerical study, in which both well-known and new sets of bases are considered in conjunction with either the standard or modified Hu–Washizu formulations. The presence of checkerboard modes is demonstrated, as is the superior performance of the modified formulation in those circumstances in which the bases chosen satisfy the conditions for well-posedness.

The structure of the rest of this work is as follows. Section 2 is devoted to the continuous problem, in its original and modified settings, and to a discussion of the relevant saddlepoint and weak problems. In Section 3 the discrete formulations are presented. The results on well-posedness established in [23] are briefly summarized, after which the conditions for equivalence between the classical and modified Hu–Washizu formulations and various other mixed and enhanced strain formulations is made clear. Section 4 is devoted to a presentation and discussion of numerical results.

In all cases the displacements are approximated by piecewise continuous bilinear functions, and five sets of bases, based on different choices of the spaces S_h and D_h for stress and strain, are constructed. Included in these choices are the assumed stress basis of Pian and Sumihara [26], and the bases corresponding to the methods of enhanced assumed strains and mixed enhanced strains. The sets are chosen in such a manner as to illustrate how certain well-known methods fit into the present theory, and some choices serve also to illustrate the problems that occur when the conditions for stability are not satisfied. The numerical results in all cases reflect the good performance predicted by the theory, for those bases that satisfy the conditions for stability, and they illustrate also in particular cases the poor coarse-mesh accuracy and volumetric locking behavior that characterises bases which fail to meet the requirements for well-posedness. Finally, the checkerboard modes predicted by the theory are illustrated in the numerical examples.

2. The boundary value problem of elasticity

In the context of elasticity, vector- and tensor- or matrix-valued functions will be written in boldface form. The scalar product of two tensors or matrices $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ will be denoted by $\boldsymbol{\sigma}:\boldsymbol{\tau}$, and is given by $\boldsymbol{\sigma}:\boldsymbol{\tau} = \sigma_{ij}\tau_{ij}$ the summation convention on repeated indices being invoked.

Consider a plane homogeneous isotropic linear elastic material body which occupies a bounded domain Ω in $\mathbb{R}^2$ with Lipschitz boundary Γ. For a prescribed body force $\boldsymbol{f}$, the governing equilibrium equation in Ω reads

$$-\operatorname{div}\boldsymbol{\sigma} = \boldsymbol{f}, \tag{2.1}$$

where $\boldsymbol{\sigma}$ is the symmetric Cauchy stress tensor. The infinitesimal strain tensor $\boldsymbol{d}$ is defined as a function of the displacement $\boldsymbol{u}$ by

$$\boldsymbol{d} = \epsilon(\boldsymbol{u}) := \tfrac{1}{2}(\nabla\boldsymbol{u} + [\nabla\boldsymbol{u}]^t). \tag{2.2}$$

The displacement is assumed to satisfy the homogeneous Dirichlet boundary condition

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma. \tag{2.3}$$

With the fourth-order elasticity tensor denoted by $\mathscr{C}$, the constitutive equation reads

$$\boldsymbol{\sigma} = \mathscr{C}\boldsymbol{d} := \lambda(\operatorname{tr}\boldsymbol{d})\mathbf{1} + 2\mu\boldsymbol{d}. \tag{2.4}$$

Here $\mathbf{1}$ is the identity tensor, $\operatorname{tr}\boldsymbol{d}$ denotes the trace of the tensor or matrix $\boldsymbol{d}$, and λ and μ are the Lamé parameters, which are constant in view of the assumption of a homogeneous body, and which are assumed positive. The parameter μ is the shear modulus. It is useful also to relate these moduli to the pair (E, ν), where E is Young's modulus and ν is Poisson's ratio; we have

$$E = \frac{\mu(2\mu + 3\lambda)}{\mu + \lambda}, \qquad \nu = \frac{\lambda}{2(\mu + \lambda)}. \tag{2.5}$$

Of particular interest is the incompressible limit, which corresponds to $\lambda \to \infty$ or $\nu \to 1/2$.

The inverse $\mathscr{C}^{-1}$ of $\mathscr{C}$ is given by

$$\boldsymbol{d} = \mathscr{C}^{-1}\boldsymbol{\sigma} = \frac{1}{2\mu}(\boldsymbol{\sigma} - \gamma(\mathrm{tr}\,\boldsymbol{\sigma})\mathbf{1}), \tag{2.6}$$

where for plane problems,

$$\gamma := \lambda/\kappa, \quad \kappa := 2(\mu + \lambda). \tag{2.7}$$

We note in particular that κ, which is defined to be twice the conventional bulk modulus, relates the volumetric parts of the stress and strain to each other through

$$\mathrm{tr}\,\boldsymbol{\sigma} = \kappa\,\mathrm{tr}\,\boldsymbol{d}. \tag{2.8}$$

Function spaces. We will make use of the space $L^2(\Omega)$ of square-integrable functions defined on Ω with the inner product and norm being denoted by $(\cdot,\cdot)_0$ and $\|\cdot\|_0$, respectively. We will also make use of the Sobolev spaces $H^m(\Omega)$, for nonnegative integers m. These are Hilbert spaces with inner product and associated norm

$$(u, v)_m := \int_\Omega \sum_{|\boldsymbol{\alpha}| \leqslant m} D^{\boldsymbol{\alpha}} u(\boldsymbol{x}) D^{\boldsymbol{\alpha}} v(\boldsymbol{x})\,\mathrm{d}x \quad \text{and} \quad \|v\|_m := (v, v)_m^{1/2}, \tag{2.9}$$

using the conventional multi-index notation. The semi-norm $|\cdot|_m$ on $H^m(\Omega)$ is defined by

$$|v|_m^2 := \int_\Omega \sum_{|\boldsymbol{\alpha}| = m} D^{\boldsymbol{\alpha}} v(\boldsymbol{x}) D^{\boldsymbol{\alpha}} v(\boldsymbol{x})\,\mathrm{d}x. \tag{2.10}$$

The space $H_0^1(\Omega)$ consists of functions in $H^1(\Omega)$ which vanish on the boundary in the sense of traces.

For the weak or variational formulations, we will require the space $V := [H_0^1(\Omega)]^2$ of displacements; this is a Hilbert space with inner product $(\cdot,\cdot)_1$ and norm $\|\cdot\|_1$ defined in the standard way; that is, $(\boldsymbol{u}, \boldsymbol{v})_1^2 = \sum_{i=1}^2 (u_i, v_i)_1$, with the norm being induced by this inner product.

The space of stresses is denoted by S, while the space of strains is denoted by D. For the continuous case these spaces are equal, and $D := \{\boldsymbol{e} | e_{ji} = e_{ij}, e_{ij} \in L^2(\Omega)\} =: S$, with norm $\|\cdot\|_0$ generated in the standard way by the L^2-norm. We also introduce the space S_0 defined by

$$S_0 := \{\boldsymbol{\tau} \in S | (\boldsymbol{\tau}, \mathbf{1})_0 = 0\}; \tag{2.11}$$

this is a closed subspace of S.

In what follows we will require the L^2-orthogonal decomposition of S into its deviatoric and spherical parts. Define the L^2-orthogonal projections sph and dev on S by

$$\mathrm{sph}\,\boldsymbol{\tau} := (1/2)(\mathrm{tr}\,\boldsymbol{\tau})\mathbf{1}, \qquad \mathrm{dev}\,\boldsymbol{\tau} := \boldsymbol{\tau} - \mathrm{sph}\,\boldsymbol{\tau}. \tag{2.12}$$

We note that $\mathrm{dev}\,S$ is a proper subset of S_0.

Rather than present the different weak formulations directly, we proceed instead by starting with the minimization or saddlepoint problems associated with the different formulations, in order to make more explicit the manner in which these are related to each other. The weak problems are then easily obtained from the necessary conditions for a minimum or saddlepoint.

The standard displacement-based variational formulation of the boundary value problem for linear elasticity may be obtained as the solution to the minimization problem of finding $\boldsymbol{u} \in V$ such that

$$I(\boldsymbol{u}) = \inf_{\boldsymbol{v} \in V} I(\boldsymbol{v}), \tag{2.13}$$

in which

$$I(\boldsymbol{v}) := \tfrac{1}{2} A(\boldsymbol{v}, \boldsymbol{v}) - \ell(\boldsymbol{v}). \tag{2.14}$$

The bilinear form $A(\cdot,\cdot)$ and linear functional $\ell(\cdot)$ are defined by

$$\begin{aligned} &A : V \times V \to \mathbb{R}, \quad A(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \mathscr{C}\boldsymbol{\epsilon}(\boldsymbol{u}) : \boldsymbol{\epsilon}(\boldsymbol{v})\,\mathrm{d}x, \\ &\ell : V \to \mathbb{R}, \quad \ell(\boldsymbol{v}) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\,\mathrm{d}x. \end{aligned} \tag{2.15}$$

From the necessary condition for a minimum one obtains the weak problem in the following form: given $\ell \in V'$, find $\boldsymbol{u} \in V$ that satisfies

$$A(\boldsymbol{u}, \boldsymbol{v}) = \ell(\boldsymbol{v}), \quad \boldsymbol{v} \in V. \tag{2.16}$$

The assumptions on $\mathscr{C}$ guarantee that $A(\cdot,\cdot)$ is symmetric, continuous, and V-elliptic, and that the problem (2.16) or (2.13) has a unique solution. However, the continuity constant depends on λ, and we therefore cannot declare that finite element solutions based on this formulation converge uniformly with respect to λ. To achieve uniform convergence it is necessary to work within the framework of mixed formulations.

Mixed formulations. The problem described above may be cast in a variety of alternative mixed forms, the term 'mixed' carrying in this context the connotation that the resulting weak formulation has a link to a saddlepoint problem. We focus on the Hu–Washizu formulation, in which the displacement, strain, and stress are unknown variables. The functional corresponding to this formulation is given by

$$\begin{aligned} J_1((\boldsymbol{v}, \boldsymbol{e}), \boldsymbol{\tau}) &:= \int_\Omega \left[\frac{1}{2}\mathscr{C}\boldsymbol{e} : \boldsymbol{e} + (\boldsymbol{\epsilon}(\boldsymbol{v}) - \boldsymbol{e}) : \boldsymbol{\tau}\right] \mathrm{d}x - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, \mathrm{d}x \\ &= \int_\Omega \left[\frac{1}{2}\lambda(\operatorname{tr}\boldsymbol{e})^2 + \mu|\boldsymbol{e}|^2 + (\boldsymbol{\epsilon}(\boldsymbol{v}) - \boldsymbol{e}) : \boldsymbol{\tau}\right] \mathrm{d}x - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, \mathrm{d}x \end{aligned} \tag{2.17}$$

and the variational problem becomes one in which we seek $(\boldsymbol{u}, \boldsymbol{d}, \boldsymbol{\sigma})$ in $V \times D \times S_0$ such that

$$J_1((\boldsymbol{u}, \boldsymbol{d}), \boldsymbol{\sigma}) = \inf_{(\boldsymbol{v},\boldsymbol{e}) \in V \times D} \sup_{\boldsymbol{\tau} \in S_0} J_1((\boldsymbol{v}, \boldsymbol{e}), \boldsymbol{\tau}). \tag{2.18}$$

It is shown in [23] that it is necessary to restrict the space of stresses to S_0 in order to show well-posedness. Equivalently, this corresponds to the weak problem of finding $\boldsymbol{u} \in V$, $\boldsymbol{d} \in D$ and $\boldsymbol{\sigma} \in S_0$ that satisfy

$$\begin{aligned} &\int_\Omega (\mathscr{C}\boldsymbol{d} - \boldsymbol{\sigma}) : \boldsymbol{e} \, \mathrm{d}x = 0, \quad \boldsymbol{e} \in D, \\ &\int_\Omega (\boldsymbol{\epsilon}(\boldsymbol{u}) - \boldsymbol{d}) : \boldsymbol{\tau} \, \mathrm{d}x = 0, \quad \boldsymbol{\tau} \in S_0, \\ &\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\epsilon}(\boldsymbol{v}) \, \mathrm{d}x = \ell(\boldsymbol{v}), \quad \boldsymbol{v} \in V. \end{aligned} \tag{2.19}$$

The three equations in (2.19) are the weak forms of, respectively, the constitutive equation, the strain–displacement equation, and the equation of equilibrium. By rearrangement, this set of equations may be cast as the standard mixed problem of finding $(\boldsymbol{u}, \boldsymbol{d}, \boldsymbol{\sigma}) \in V \times D \times S_0$ that satisfy

$$\begin{aligned} &a_1((\boldsymbol{u}, \boldsymbol{d}), (\boldsymbol{v}, \boldsymbol{e})) + b_1((\boldsymbol{v}, \boldsymbol{e}), \boldsymbol{\sigma}) = \ell(\boldsymbol{v}), \quad (\boldsymbol{v}, \boldsymbol{e}) \in V \times D, \\ &b_1((\boldsymbol{u}, \boldsymbol{d}), \boldsymbol{\tau}) = 0, \quad \boldsymbol{\tau} \in S_0, \end{aligned} \tag{2.20}$$

where

$$\begin{aligned} &a_1((\boldsymbol{u}, \boldsymbol{d}), (\boldsymbol{v}, \boldsymbol{e})) := (\mathscr{C}\boldsymbol{d}, \boldsymbol{e})_0, \\ &b_1(\boldsymbol{v}, \boldsymbol{e}, \boldsymbol{\tau}) := (\boldsymbol{\epsilon}(\boldsymbol{v}) - \boldsymbol{e}, \boldsymbol{\tau})_0. \end{aligned} \tag{2.21}$$

The functional J_1 may thus be written in the more compact form

$$J_1((\boldsymbol{v}, \boldsymbol{e}), \boldsymbol{\tau}) = \tfrac{1}{2}a_1((\boldsymbol{v}, \boldsymbol{e}), (\boldsymbol{v}, \boldsymbol{e})) + b_1((\boldsymbol{v}, \boldsymbol{e}), \boldsymbol{\tau}) - \ell(\boldsymbol{v}). \tag{2.22}$$

It is readily shown [23] that the problem (2.20) has a unique solution $(\boldsymbol{u}, \boldsymbol{d}, \boldsymbol{\sigma})$ in $V \times D \times S_0$, which moreover satisfies the bound

$$\|\boldsymbol{u}\|_1 + \|\boldsymbol{d}\|_0 + \|\boldsymbol{\sigma}\|_0 \leqslant c(\lambda)\|\ell\|_{V'}. \tag{2.23}$$

The constant in the bound is λ-dependent, and so we are not able to obtain stability of the solution in the incompressible limit through this direct approach. This motivates the need to consider alternative formulations of the Hu–Washizu problem.

We do so by first constraining the space in which a solution is sought, to the subspace $(D \times S_0)_{\mathrm{tr}}$ of $D \times S_0$ consisting of those functions that satisfy the volumetric relation (2.8); that is,

$$(D \times S_0)_{\mathrm{tr}} = \{(\boldsymbol{e}, \boldsymbol{\tau}) \in D \times S_0 | \operatorname{tr}\boldsymbol{\tau} = \kappa \operatorname{tr}\boldsymbol{e} \text{ a.e. in } \Omega\}. \tag{2.24}$$

This constraint is imposed directly on the integrand of (2.17) to eliminate the term $\operatorname{tr}\boldsymbol{e}$. Since

$$\tfrac{1}{2}\boldsymbol{e} : \mathscr{C}\boldsymbol{e} = \tfrac{1}{2}\lambda(\operatorname{tr}\boldsymbol{e})^2 + \mu|\boldsymbol{e}|^2 = (1/4)\kappa(\operatorname{tr}\boldsymbol{e})^2 + \mu|\operatorname{dev}\boldsymbol{e}|^2,$$

we obtain the problem of finding $(\boldsymbol{u},\boldsymbol{d},\boldsymbol{\sigma}) \in V \times (D \times S_0)_{\mathrm{tr}}$ such that

$$\widetilde{J}(\boldsymbol{u},\boldsymbol{d},\boldsymbol{\sigma}) = \inf_{\boldsymbol{v},\boldsymbol{e}} \sup_{\boldsymbol{\tau}} \{\widetilde{J}(\boldsymbol{v},\boldsymbol{e},\boldsymbol{\tau}) | \boldsymbol{v} \in V,\ (\boldsymbol{e},\boldsymbol{\tau}) \in (D \times S_0)_{\mathrm{tr}}\}, \tag{2.25}$$

in which the new functional $\widetilde{J}$ is given by

$$\widetilde{J}((\boldsymbol{v},\boldsymbol{e}),\boldsymbol{\tau}) := \int_\Omega \left[\mu|\mathrm{dev}\,\boldsymbol{e}|^2 + (\boldsymbol{\epsilon}(\boldsymbol{v}) - \mathrm{dev}\,\boldsymbol{e}) : \boldsymbol{\tau} - \frac{1}{4\kappa}(\mathrm{tr}\,\boldsymbol{\tau})^2\right] \mathrm{d}x - \ell(\boldsymbol{v}). \tag{2.26}$$

If we define the bilinear forms

$$\begin{aligned}
&\widetilde{a}((\boldsymbol{u},\boldsymbol{d}),(\boldsymbol{v},\boldsymbol{e})) := 2\mu(\mathrm{dev}\,\boldsymbol{d}, \mathrm{dev}\,\boldsymbol{e})_0,\\
&\widetilde{b}((\boldsymbol{v},\boldsymbol{e}),\boldsymbol{\sigma}) := (\boldsymbol{\epsilon}(\boldsymbol{v}) - \mathrm{dev}\,\boldsymbol{e}, \boldsymbol{\sigma})_0,\\
&c(\boldsymbol{\sigma},\boldsymbol{\tau}) := (\mathrm{tr}\,\boldsymbol{\sigma}, \mathrm{tr}\,\boldsymbol{\tau})_0,
\end{aligned} \tag{2.27}$$

then the weak problem corresponding to the saddlepoint problem (2.25) is that of finding $(\boldsymbol{u},\boldsymbol{d},\boldsymbol{\sigma}) \in V \times (D \times S_0)_{\mathrm{tr}}$ that satisfy

$$\begin{aligned}
&\widetilde{a}((\boldsymbol{u},\boldsymbol{d}),(\boldsymbol{v},\boldsymbol{e})) + \widetilde{b}((\boldsymbol{v},\boldsymbol{e}),\boldsymbol{\sigma}) = \ell(\boldsymbol{v}),\\
&\widetilde{b}((\boldsymbol{u},\boldsymbol{d}),\boldsymbol{\tau}) - (1/2\kappa)c(\boldsymbol{\sigma},\boldsymbol{\tau}) = 0,
\end{aligned} \tag{2.28}$$

for all $(\boldsymbol{v},\boldsymbol{e},\boldsymbol{\tau}) \in V \times (D \times S_0)_{\mathrm{tr}}$. Thus the effect of this transformation is to concentrate the volumetric behavior entirely in the term corresponding to the bilinear form $c(\cdot,\cdot)$.

While the modified Hu–Washizu formulation in the form (2.28) is a good candidate formulation in which to seek continuous and discrete solutions that are well-posed in the incompressible limit, it is seen that this problem is posed on the subspace of functions that satisfy the volumetric part of the constitutive equation. Indeed, one obtains from Eq. $(2.28)_1$ the equilibrium equation, by setting $\boldsymbol{e} = \boldsymbol{0}$, and then, by setting $\boldsymbol{v} = \boldsymbol{0}$, the deviatoric part of the elasticity equation, that is,

$$\mathrm{dev}\,\boldsymbol{\sigma} = 2\mu\,\mathrm{dev}\,\boldsymbol{d}.$$

The second of equations (2.28) gives, by setting $\boldsymbol{\tau} = \boldsymbol{\tau}^{\mathrm{D}}$, a deviatoric tensor,

$$\mathrm{dev}\,\boldsymbol{d} = \mathrm{dev}\,\boldsymbol{\epsilon}(\boldsymbol{u})$$

and finally, by setting $\boldsymbol{\tau} = \boldsymbol{\tau}^{\mathrm{S}}$, a spherical tensor,

$$\mathrm{tr}\,\boldsymbol{\sigma} = \kappa\,\mathrm{tr}\,\boldsymbol{\epsilon}(\boldsymbol{u}) = \mathrm{div}\,\boldsymbol{u}.$$

The final equation, viz. $\mathrm{tr}\,\boldsymbol{d} = \mathrm{div}\,\boldsymbol{u}$, is deduced by using the function space setting of the problem.

When considering finite element approximations, it is preferable to be able to pose the problem on the original spaces V, D, and S_0, rather than on subspaces of these, in order to be able to make use of conventional finite element bases. To this end, and still with a view to modifying the Hu–Washizu formulation in such a way as to obtain uniform well-posedness, we return to the functional (2.17) and introduce the volumetric relation partially, in order to eliminate only its appearance in the term involving λ. This then leads to the modified functional J_0 defined by

$$\begin{aligned}
J_0(\boldsymbol{v},\boldsymbol{e},\boldsymbol{\tau}) &:= \int_\Omega \left[\mu\boldsymbol{e} : \boldsymbol{e} + (\boldsymbol{\epsilon}(\boldsymbol{v}) - \boldsymbol{e} + \gamma(\mathrm{tr}\,\boldsymbol{e})\boldsymbol{1}) : \boldsymbol{\tau} - \tfrac{1}{2}(\gamma^2/\lambda)(\mathrm{tr}\,\boldsymbol{\tau})^2\right] \mathrm{d}x - \ell(\boldsymbol{v})\\
&= \int_\Omega \left[\mu\boldsymbol{e} : \boldsymbol{e} + (\boldsymbol{\epsilon}(\boldsymbol{v}) - 2\mu\mathscr{C}^{-1}\boldsymbol{e}) : \boldsymbol{\tau} - \tfrac{1}{2}(\gamma^2/\lambda)(\mathrm{tr}\,\boldsymbol{\tau})^2\right] \mathrm{d}x - \ell(\boldsymbol{v}).
\end{aligned} \tag{2.29}$$

The saddlepoint problem of minimizing this functional over the spaces of displacements and strains, and maximizing over the stresses, leads to the weak problem of finding $(\boldsymbol{u},\boldsymbol{d},\boldsymbol{\sigma}) \in V \times D \times S_0$ such that

$$\begin{aligned}
&a_0((\boldsymbol{u},\boldsymbol{d}),(\boldsymbol{v},\boldsymbol{e})) + b_0((\boldsymbol{v},\boldsymbol{e}),\boldsymbol{\sigma}) = \ell(\boldsymbol{v}), \quad (\boldsymbol{v},\boldsymbol{e}) \in V \times D,\\
&b_0((\boldsymbol{u},\boldsymbol{d}),\boldsymbol{\tau}) - (\gamma^2/\lambda)c(\boldsymbol{\sigma},\boldsymbol{\tau}) = 0, \quad \boldsymbol{\tau} \in S_0,
\end{aligned} \tag{2.30}$$

in which the bilinear forms are defined by

$$\begin{aligned}
&a_0((\boldsymbol{u},\boldsymbol{d}),(\boldsymbol{v},\boldsymbol{e})) := 2\mu(\boldsymbol{d},\boldsymbol{e})_0,\\
&b_0((\boldsymbol{v},\boldsymbol{e}),\boldsymbol{\sigma}) := (\boldsymbol{\epsilon}(\boldsymbol{v}) - 2\mu\mathscr{C}^{-1}\boldsymbol{e},\boldsymbol{\sigma})_0
\end{aligned} \tag{2.31}$$

and $c(\cdot,\cdot)$ is as defined before. It is important to note that the desired solution is found by seeking a saddlepoint over the original spaces, in contrast to the situation with the functional (2.26).

Now it is possible to formulate a family of problems, parametrized by a scalar α, by interpolating between the functionals J_0 and J_1 in such a way that the cases $\alpha = 0$ and $\alpha = 1$ correspond to the problems involving the functional J_0 and J_1, whose solutions of course coincide in the continuous case. We do this by adding to the functional J_0 the term

$$\alpha\left[\tfrac{1}{2}\lambda(\operatorname{tr}\boldsymbol{e})^2 - \gamma(\operatorname{tr}\boldsymbol{e})(\operatorname{tr}\boldsymbol{\tau}) - \tfrac{1}{2}(\gamma^2/\lambda)(\operatorname{tr}\boldsymbol{\tau})^2\right],$$

for some α, which is identically zero for all $\boldsymbol{e}$ and $\boldsymbol{\tau}$ lying in the subspace $(D \times S_0)_{\mathrm{tr}}$. We note that it is not necessary to constrain α to lie in the range $[0, 1]$, and we do not impose this restriction. This leads to the functional

$$J_\alpha(\boldsymbol{v}, \boldsymbol{e}, \boldsymbol{\tau}) := \int_\Omega \left[\tfrac{1}{2}\alpha\lambda(\operatorname{tr}\boldsymbol{e})^2 + \mu|\boldsymbol{e}|^2 + (\boldsymbol{\epsilon}(\boldsymbol{v}) - 2\mu\mathscr{C}^{-1}\boldsymbol{e} - \alpha\gamma(\operatorname{tr}\boldsymbol{e})\mathbf{1}) : \boldsymbol{\tau} - \tfrac{1}{2}(1-\alpha)(\gamma^2/\lambda)(\operatorname{tr}\boldsymbol{\tau})^2\right] \mathrm{d}x - \ell(\boldsymbol{v}). \tag{2.32}$$

The problem of seeking a saddlepoint is then equivalent to that of finding $(\boldsymbol{u}, \boldsymbol{d}, \boldsymbol{\sigma}) \in V \times D \times S_0$ that satisfy

$$\begin{aligned} &a_\alpha((\boldsymbol{u}, \boldsymbol{d}), (\boldsymbol{v}, \boldsymbol{e})) + b_\alpha((\boldsymbol{v}, \boldsymbol{e}), \boldsymbol{\sigma}) = \ell(\boldsymbol{v}), \quad (\boldsymbol{v}, \boldsymbol{e}) \in V \times D, \\ &b_\alpha((\boldsymbol{u}, \boldsymbol{d}), \boldsymbol{\tau}) - (1-\alpha)(\gamma^2/\lambda)c(\boldsymbol{\sigma}, \boldsymbol{\tau}) = 0, \quad \boldsymbol{\tau} \in S_0, \end{aligned} \tag{2.33}$$

where the bilinear forms are given by

$$\begin{aligned} &a_\alpha((\boldsymbol{u}, \boldsymbol{d}), (\boldsymbol{v}, \boldsymbol{e})) := 2\mu(\boldsymbol{d}, \boldsymbol{e})_0 + \alpha\lambda(\operatorname{tr}\boldsymbol{d}, \operatorname{tr}\boldsymbol{e})_0, \\ &b_\alpha((\boldsymbol{v}, \boldsymbol{e}), \boldsymbol{\sigma}) := (\boldsymbol{\epsilon}(\boldsymbol{v}) - 2\mu\mathscr{C}^{-1}\boldsymbol{e}, \boldsymbol{\sigma})_0 - \alpha\gamma(\operatorname{tr}\boldsymbol{\sigma}, \operatorname{tr}\boldsymbol{e})_0. \end{aligned} \tag{2.34}$$

This is the modified problem introduced in [23], albeit without the link to an appropriate saddlepoint problem.

The bilinear forms can be expressed in terms of the forms corresponding to the standard Hu–Washizu problem (2.20), which is also problem (2.33) with $\alpha = 1$, according to

$$\begin{aligned} &a_\alpha((\boldsymbol{u}, \boldsymbol{d}), (\boldsymbol{v}, \boldsymbol{e})) = a_1((\boldsymbol{u}, \boldsymbol{d}), (\boldsymbol{v}, \boldsymbol{e})) + (\alpha - 1)\lambda c(\boldsymbol{d}, \boldsymbol{e}), \\ &b_\alpha((\boldsymbol{v}, \boldsymbol{e}), \boldsymbol{\tau}) = b_1((\boldsymbol{v}, \boldsymbol{e}), \boldsymbol{\tau}) + (\alpha - 1)\gamma c(\boldsymbol{e}, \boldsymbol{\tau}). \end{aligned} \tag{2.35}$$

The solution to the original Hu–Washizu problem may be recovered from the modified problem, and the solution is bounded in the incompressible limit.

Lemma 2.1 [23]. *For $\alpha \neq -\mu/\lambda$, there exists a unique solution $(\boldsymbol{u}, \boldsymbol{d}, \boldsymbol{\sigma}) \in V \times D \times S_0$ of the modified Hu–Washizu formulation (2.33). Moreover, the solution does not depend on α, and satisfies the bound*

$$\|\boldsymbol{u}\|_1 + \|\boldsymbol{d}\|_0 + \|\boldsymbol{\sigma}\|_0 \leqslant C\|\ell\|_{V'},$$

in which the constant C is independent of λ.

3. Finite element approximations

3.1. Finite element spaces

We are concerned with finite element approximations based on quasi-uniform, shape regular meshes of quadrilaterals on polygonal domains in $\mathbb{R}^2$. The finite element meshes are defined by maps from a reference square $\widehat{K} = (-1, 1)^2$.

For nonnegative integer k, let $\mathscr{P}_k(\cdot)$ denote the space of polynomials in two variables of total degree less than or equal to k, and $\mathscr{Q}_k(\cdot)$ the space of polynomials in two variables of total degree less than or equal to k in each variable. A typical element $K \in \mathscr{T}_h$ is generated by an isoparametric map F_K from the reference element $\widehat{K}$, in which F_K is defined using the basis functions corresponding to $\mathscr{Q}_1$. It is clear that if $\hat{v} \in \mathscr{Q}_1(\widehat{K})$, then $\hat{v} \circ F_K^{-1}$ is in general not a polynomial on the quadrilateral K.

The analysis in [23] is carried out for the case of meshes of affine-equivalent elements. Numerical results will also be presented for the more general case of meshes of arbitrary quadrilaterals, in Section 5.

The finite element space of displacements is taken to be the space of continuous functions whose restrictions to an element K are obtained by maps of *bilinear* functions from the reference element; that is,

$$V_h := \left\{\boldsymbol{v}_h \in V \cap C(\overline{\Omega}), \boldsymbol{v}_h|_K = \hat{\boldsymbol{v}}_h \circ F_K^{-1}, \hat{\boldsymbol{v}}_h \in \mathscr{Q}_1(\widehat{K})^2 \text{ for all } K \in \mathscr{T}_h\right\}. \tag{3.1}$$

The spaces of stresses and strains are discretized by defining the finite-dimensional spaces

$$\begin{aligned} &S_h := \left\{\boldsymbol{\tau}_h \in S_0 | (\boldsymbol{\tau}_h|_K)_{ij} = (\hat{\boldsymbol{\tau}}_h)_{ij} \circ F_K^{-1}, \hat{\boldsymbol{\tau}}_h \in S_\square \text{ for all } K \in \mathscr{T}_h\right\}, \\ &D_h := \left\{\boldsymbol{e}_h \in S_0 | (\boldsymbol{e}_h|_K)_{ij} = (\hat{\boldsymbol{e}}_h)_{ij} \circ F_K^{-1}, \hat{\boldsymbol{e}}_h \in D_\square \text{ for all } K \in \mathscr{T}_h\right\}, \end{aligned} \tag{3.2}$$

where $D_\Box$ and $S_\Box$ are the reference bases of strains and stresses, defined on $\widehat{K}$. These two variables are defined locally on each element and no continuity conditions apply at the element boundaries.

Given the problems associated with establishing well-posedness of the continuous classical Hu–Washizu formulation in the incompressible limit, we go directly to the discrete version of problem (2.33), which takes the following form: find $(\boldsymbol{u}_h, \boldsymbol{d}_h, \boldsymbol{\sigma}_h) \in V_h \times D_h \times S_h$ such that

$$\begin{aligned} &a_\alpha((\boldsymbol{u}_h, \boldsymbol{d}_h), (\boldsymbol{v}_h, \boldsymbol{e}_h)) + b_\alpha((\boldsymbol{v}_h, \boldsymbol{e}_h), \boldsymbol{\sigma}_h) = \ell(\boldsymbol{v}_h), \quad (\boldsymbol{v}_h, \boldsymbol{e}_h) \in V_h \times D_h, \\ &b_\alpha((\boldsymbol{u}_h, \boldsymbol{d}_h), \boldsymbol{\tau}_h) - (1-\alpha)(\gamma^2/\lambda) c(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) = 0, \quad \boldsymbol{\tau}_h \in S_h. \end{aligned} \tag{3.3}$$

Since we will be making frequent reference to the classical Hu–Washizu formulation, which corresponds to (3.3) with $\alpha = 1$, we display this problem explicitly, in the discrete analogue of (2.19): we are required to find $(\boldsymbol{u}_h, \boldsymbol{d}_h, \boldsymbol{\sigma}_h) \in V_h \times D_h \times S_h$ that satisfy

$$\begin{aligned} &\int_\Omega (\mathscr{C}\boldsymbol{d}_h - \boldsymbol{\sigma}_h) : \boldsymbol{e}_h \, \mathrm{d}x = 0, \quad \boldsymbol{e}_h \in D_h, \\ &\int_\Omega (\boldsymbol{\epsilon}(\boldsymbol{u}_h) - \boldsymbol{d}_h) : \boldsymbol{\tau}_h \, \mathrm{d}x = 0, \quad \boldsymbol{\tau}_h \in S_h, \\ &\int_\Omega \boldsymbol{\sigma}_h : \boldsymbol{\epsilon}(\boldsymbol{v}_h) \, \mathrm{d}x = \ell(\boldsymbol{v}_h), \quad \boldsymbol{v}_h \in V_h. \end{aligned} \tag{3.4}$$

We conclude this section with a result on the relationship between the different formulations that arise from (3.3) for various choices of α. The following is shown in [23].

Lemma 3.1. *Assume that*

$$S_h \subset D_h; \tag{3.5}$$

$$\operatorname{tr} D_h \mathbf{1} \subset D_h. \tag{3.6}$$

Then the solution of the problem (3.3) *does not depend on* α.

3.2. Examples of bases

Before giving some concrete examples of bases for S_h and D_h, we recall the Voigt representation of the tensorial quantities stress and strain in vectorial form, in two dimensions. These are written as

$$\boldsymbol{d} = \begin{bmatrix} d_{11} \\ d_{22} \\ 2d_{12} \end{bmatrix} \quad \text{and} \quad \boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}. \tag{3.7}$$

The spaces S_h and D_h will be generated from bases defined on $\widehat{K}$, and we will make use of the following bases on $\widehat{K}$:

$$\mathrm{Id} := \operatorname{span} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A := \operatorname{span} \begin{bmatrix} \eta & 0 \\ 0 & \xi \\ 0 & 0 \end{bmatrix}, \quad B := \operatorname{span} \begin{bmatrix} \xi & 0 \\ 0 & \eta \\ 0 & 0 \end{bmatrix}, \quad C := \operatorname{span} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \xi & \eta \end{bmatrix}. \tag{3.8}$$

The basis $\mathrm{Id} + A$ was introduced by Pian and Sumihara [26] in their assumed stress formulation, to approximate stresses, while $\mathrm{Id} + A + B$ has been used by Kasper and Taylor [21,22] in their mixed enhanced strain formulation for the approximation of D_h.

We remark that $\operatorname{tr}(\mathrm{Id} + A) = \operatorname{tr}(\mathrm{Id} + A + B) = P_1(\widehat{K})$ and $\operatorname{tr}(\mathrm{Id} + C) = P_0(\widehat{K})$.

Of special interest will be the choices given in Table 1. We note that Cases II, III and V satisfy the assumptions for equivalence between the classical and modified Hu–Washizu formulations, but that Cases I and IV do not satisfy the second of these.

Table 1
Examples of bases for S_h and D_h

Case	S_h	D_h	Equivalence
I	$\mathrm{Id} + A$	$\mathrm{Id} + A$	HW $\neq$ MHW
II	$\mathrm{Id} + A$	$\mathrm{Id} + A + B$	HW $=$ MHW
III	$\mathrm{Id} + C$	$\mathrm{Id} + C$	HW $=$ MHW
IV	$\mathrm{Id} + A + C$	$\mathrm{Id} + A + C$	HW $\neq$ MHW
V	$\mathrm{Id} + A + C$	$\mathrm{Id} + A + B + C$	HW $=$ MHW

3.3. Well-posedness and convergence

We present here a summary of the main results on well-posedness and convergence for the modified Hu–Washizu formulation, which have been established in [23]. To do so, we return to the decomposition (2.12) of the stress into its deviatoric and spherical parts. Now while this decomposition is unambiguous in the continuous case, care has to be exercised in the discrete case. To fix ideas, take Case I in Table 1. For this choice of space any stress $\boldsymbol{\sigma}_h$ has the representation on the reference element, and using Voigt notation,

$$\boldsymbol{\sigma}_h = \begin{pmatrix} a_1 + b_1\eta \\ a_2 + b_2\xi \\ a_3 \end{pmatrix}. \tag{3.9}$$

The spherical and deviatoric parts are therefore given by

$$\operatorname{sph}\boldsymbol{\sigma}_h = \frac{1}{2}[(a_1+a_2) + b_1\eta + b_2\xi]\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \qquad \operatorname{dev}\boldsymbol{\sigma}_h = \frac{1}{2}\begin{pmatrix} (a_1-a_2)+(b_1\eta - b_2\xi) \\ -(a_1-a_2)-(b_1\eta-b_2\xi) \\ a_3 \end{pmatrix}. \tag{3.10}$$

It follows that:

$$\operatorname{dev} S_h = \operatorname{span}\begin{pmatrix} 1 & 0 & \eta & \xi \\ -1 & 0 & -\eta & -\xi \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

which is *not* a subspace of S_h. In view of the special role played by the spherical part of the stress in determining well-posedness in the incompressible limit, and hence of the deviatoric part, it is essential to work with a decomposition in which the two subspaces are orthogonal complements, in the conventional sense. For this purpose the discrete deviatoric operator dev_h has been introduced. This is defined to be the orthogonal projection onto S_h of the deviatoric part of $\boldsymbol{\sigma}_h$; that is,

$$\operatorname{dev}_h\boldsymbol{\sigma}_h = P_{S_h}\operatorname{dev}\boldsymbol{\sigma}_h, \tag{3.11}$$

where P_{S_h} denotes the L^2-orthogonal projection operator onto S_h. Then the discrete spherical operator sph_h is defined in such a way that $\operatorname{sph}_h S_h$ is the orthogonal complement of $\operatorname{dev}_h S_h$; that is, we have the decomposition

$$S_h = \operatorname{dev}_h S_h \oplus \operatorname{sph}_h S_h. \tag{3.12}$$

Returning to Case I, it is easily seen that

$$\operatorname{dev}_h S_h = \operatorname{span}\begin{pmatrix} 1 & 0 & \eta & 0 \\ -1 & 0 & 0 & \xi \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad \operatorname{sph}_h S_h = \operatorname{span}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

The analogue of the space of pressures in the Stokes problem is now the space $\widetilde{M}_h$, defined by

$$\widetilde{M}_h := \operatorname{tr}\operatorname{sph}_h S_h. \tag{3.13}$$

The following theorem summarises the main results obtained in [23].

Theorem 3.2. *Suppose that the spaces V_h, D_h and S_h, the bilinear forms $a_\alpha(\cdot,\cdot)$ and $b_\alpha(\cdot,\cdot)$, and the parameter α satisfy the following conditions*:

(*a*) $S_h \subset D_h$;
(*b*) $\|P_{S_h}\boldsymbol{\epsilon}(\boldsymbol{v}_h)\|_0 \geqslant c\|\boldsymbol{\epsilon}(\boldsymbol{v}_h)\|_0$ *for all* $\boldsymbol{v}_h \in V_h$, *where* $0 < c < 1$;
(*c*) $(V_h, \widetilde{M}_h)$ *forms a stable Stokes pairing*;
(*d*) α *satisfies the bounds* $\max(-\frac{c_\ell\mu}{\lambda}, 1-\frac{c_\ell}{\omega}) \leqslant \alpha \leqslant \min(\frac{C_u\mu}{\lambda}, 1)$, $0 < c_\ell < 1$, $0 < C_u < \infty$, *where* $\omega < 1$ *is the constant in the bound* $(\mathscr{C}^{-1}\boldsymbol{\tau}_h, \boldsymbol{\tau}_h)_0 \geqslant \frac{1-\omega}{2\mu}\|\boldsymbol{\tau}_h\|_0^2$, $\boldsymbol{\tau}_h \in \operatorname{dev}_h S_h$.

Then the discretization error $\eta_h^2 := \|\boldsymbol{u}-\boldsymbol{u}_h\|_1^2 + \|\boldsymbol{d}-\boldsymbol{d}_h\|_0^2 + \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_0^2$ *is bounded by the best approximation error*

$$\eta_h^2 \leqslant C\left(\inf_{\boldsymbol{v}_h\in V_h}\|\boldsymbol{u}-\boldsymbol{v}_h\|_1^2 + \inf_{\boldsymbol{e}_h\in D_h}\|\boldsymbol{d}-\boldsymbol{e}_h\|_0^2 + \inf_{\boldsymbol{\tau}_h\in S_h}\|\boldsymbol{\sigma}-\boldsymbol{\tau}_h\|_0^2\right).$$

Now for all five cases in Table 1, it is easy to show that

$$\widetilde{M}_h|_K = P_0.$$

Thus the pairing $(V_h, \widetilde{M}_h)$ is the classical $Q_1 - P_0$ pair, for which the existence of checkerboard modes is well-known (see, for example, [20]). Indeed, in the following section we will demonstrate through selected examples the existence of these checkerboard modes.

As in the case of the $Q_1 - P_0$ element, the checkerboard modes can be filtered out, and furthermore it can be shown that the displacement satisfies an a priori error estimate that is uniform with respect to λ. This is achieved by reducing the three-field formulation to one involving only the displacements, through a process of static condensation (see [33] for the details in the matrix context). This gives the problem of finding $\boldsymbol{u}_h \in V_h$ such that

$$(Q_h\boldsymbol{\epsilon}(\boldsymbol{u}_h), \mathscr{C}_h Q_h\boldsymbol{\epsilon}(\boldsymbol{v}_h))_0 = \ell(\boldsymbol{v}_h), \quad \boldsymbol{v}_h \in V_h. \tag{3.14}$$

Here Q_h is a projection operator and $\mathscr{C}_h$ is a discrete elasticity operator, which inherits the properties of symmetry and positive-definiteness of the continuous operator.

The general conditions for the error estimate in the displacement are rather technical, and are specialized here for the choices of bases in Table 1. First we carry out an orthogonal decomposition of S_h by writing $S_h = S_h^c \oplus S_h^t$, where S_h^c is the subspace of those members $\boldsymbol{\tau}_h$ of S_h for which $\operatorname{tr}\boldsymbol{\tau}_h\mathbf{1} \in S_h$, and S_h^t its orthogonal decomposition. Then we have the following result.

Lemma 3.3 [23]. *Assume that $\alpha \neq -\mu/\lambda$. For any of the choices of bases in Table 1,*

(*a*) *if $S_h \subset D_h$ and $\mathscr{C}D_h = D_h$, then*

$$\mathscr{C}_h\boldsymbol{e}_h = \mathscr{C}P_{S_h^c}\boldsymbol{e}_h + \frac{2\mu}{1-\gamma}P_{S_h^t}\boldsymbol{e}_h; \tag{3.15}$$

(*b*) *if $S_h = D_h$ and $\alpha \neq -2\mu/\lambda$, then*

$$\mathscr{C}_h\boldsymbol{e}_h = \mathscr{C}P_{S_h^c}\boldsymbol{e}_h + \theta(\alpha,\lambda,\mu)P_{S_h^t}\boldsymbol{e}_h, \tag{3.16}$$

where $\theta(\alpha,\lambda,\mu) = \dfrac{4(\mu+\lambda)^2(2\mu+\alpha\lambda)}{\lambda^2 + (2\mu+3\lambda)(2\mu+\alpha\lambda)}$.

Furthermore, for all cases in Table 1, $Q_h = P_{S_h}$, and under conditions (a) or (b), there exists a constant $C > 0$, independent of λ, such that

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_1 \leqslant Ch.$$

Note. The operators can be extended to the cases $\alpha = -\mu/\lambda$ and, for some choices of S_h, to $\alpha = -2\mu/\lambda$.

3.4. Equivalent formulations

In this section we examine the conditions for equivalence between the Hu–Washizu and other mixed and enhanced formulations. Some of these equivalence results or limitation conditions have been known for some time, while others to be presented here are new. We present in a unified format the relationships that exist between different mixed formulations, the enhanced formulations invariably derived from them, and the conditions on the finite element spaces that would ensure equivalence.

Conditions for equivalence between the standard displacement-based formulation and various mixed formulations, as well as equivalence conditions between the mixed formulations, has been a subject of study that goes back to the pioneering work of Fraeijs de Veubeke [11–13]. Stolarski and Belytschko [36] have shown that, if the spaces of stresses and strains satisfy the inclusion

$$S_h \subset \mathscr{C}D_h, \tag{3.17}$$

then the classical Hu–Washizu formulation (3.4) is equivalent to the Hellinger–Reissner problem of finding $(\boldsymbol{u}_h, \boldsymbol{\sigma}_h) \in V_h \times S_h$ such that

$$\begin{aligned} &\int_\Omega \mathscr{C}^{-1}\boldsymbol{\sigma}_h : \boldsymbol{\tau}_h\,\mathrm{d}x - \int_\Omega \boldsymbol{\epsilon}(\boldsymbol{u}_h) : \boldsymbol{\tau}_h\,\mathrm{d}x = 0, \quad \boldsymbol{\tau}_h \in S_h,\\ &\int_\Omega \boldsymbol{\epsilon}(\boldsymbol{v}_h) : \boldsymbol{\sigma}_h\,\mathrm{d}x = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}_h\,\mathrm{d}x, \quad \boldsymbol{v}_h \in V_h. \end{aligned} \tag{3.18}$$

We arrive at the equivalence result by a different route, by noting that the assumptions (3.5) and (3.6), applied to $(3.4)_1$, imply that $\boldsymbol{\sigma}_h = \mathscr{C}\boldsymbol{d}_h$. Direct substitution in the second of equations (3.4) gives (3.18).

We note that (3.5) and (3.6) yield (3.17). Thus, a solution of (3.4) with assumptions (3.5) and (3.6) is also a solution of (3.4) with the assumption (3.17).

For the converse, suppose that (3.17) is satisfied, and define the space $D_h^* := D_h + \mathscr{C}D_h$. We find that $\mathscr{C}D_h^* = D_h^*$ and thus $S_h \subset D_h^*$, and $\operatorname{tr} D_h^* \mathbf{1} \subset D_h^*$. We designate by $(\boldsymbol{u}_h^*, \boldsymbol{d}_h^*, \boldsymbol{\sigma}_h^*) \in V_h \times D_h^* \times S_h$ the solution to (3.4) with this choice of spaces. But since this formulation also reduces to a Hellinger–Reissner problem posed on $V_h \times S_h$, it follows that $\boldsymbol{u}_h^* = \boldsymbol{u}_h$ and $\boldsymbol{\sigma}_h^* = \boldsymbol{\sigma}_h$. Finally, we have $\boldsymbol{d}_h^* = \mathscr{C}^{-1}\boldsymbol{\sigma}_h^* = \mathscr{C}^{-1}\boldsymbol{\sigma}_h = \boldsymbol{d}_h$.

The method of *mixed enhanced strains* has been derived in [21,22] starting from the Hu–Washizu formulation. We present the derivation here in a form that sheds light on the structure of the enhanced formulation, which will in turn make it simple to relate it to other mixed or enhanced methods.

Assume that (3.5) holds. Then D_h admits the L^2-orthogonal decomposition

$$D_h = S_h \oplus \widetilde{D}_h, \tag{3.19}$$

in which $\widetilde{D}_h \perp S_h$. From the first of equations (3.4), setting $\boldsymbol{e} = \boldsymbol{\tau}_h \in S_h$ we find that $(\boldsymbol{\sigma}_h - \mathscr{C}\boldsymbol{d}_h, \boldsymbol{\tau}_h)_0 = 0$ for all $\boldsymbol{\tau}_h \in S_h$, that is,

$$\boldsymbol{\sigma}_h = P_{S_h}\mathscr{C}\boldsymbol{d}_h, \tag{3.20}$$

where P_{S_h} denotes the locally defined projection onto S_h with respect to the L^2-scalar product. Using the same equation, this time setting $\boldsymbol{e}_h = \widetilde{\boldsymbol{e}}_h \in \widetilde{D}_h$, we obtain

$$\int_\Omega \mathscr{C}\boldsymbol{d}_h : \widetilde{\boldsymbol{e}}_h \,\mathrm{d}x = 0. \tag{3.21}$$

From the second of equations (3.4), we find that

$$\boldsymbol{d}_h = P_{S_h}\boldsymbol{\epsilon}(\boldsymbol{u}_h) + \widetilde{\boldsymbol{d}}_h \tag{3.22}$$

for some $\widetilde{\boldsymbol{d}}_h \in \widetilde{D}_h$. Finally, substituting these expressions in the equilibrium Eq. $(3.4)_3$, we obtain the mixed enhanced strain formulation in the following form: find $(\boldsymbol{u}_h, \widetilde{\boldsymbol{d}}_h) \in V_h \times \widetilde{D}_h$ that satisfy

$$\int_\Omega (P_{S_h}\boldsymbol{\epsilon}(\boldsymbol{v}_h) + \widetilde{\boldsymbol{e}}_h) : \mathscr{C}(P_{S_h}\boldsymbol{\epsilon}(\boldsymbol{u}_h) + \widetilde{\boldsymbol{d}}_h)\,\mathrm{d}x = \ell(\boldsymbol{v}_h), \quad (\boldsymbol{v}_h, \widetilde{\boldsymbol{e}}_h) \in V_h \times \widetilde{D}_h. \tag{3.23}$$

We thus have the following result.

Lemma 3.4. *Under the assumption* (3.5), *the Hu–Washizu and mixed enhanced strain formulations are equivalent, in the following sense: if* $(\boldsymbol{u}_h, \boldsymbol{d}_h, \boldsymbol{\sigma}_h) \in V_h \times D_h \times S_h$ *is a solution of problem* (3.4), *then* $(\boldsymbol{u}_h, \boldsymbol{d}_h - P_{S_h}\boldsymbol{\epsilon}(\boldsymbol{v}_h)) \in V_h \times \widetilde{D}_h$ *is the solution of* (3.23). *Conversely, if* $(\boldsymbol{u}_h, \widetilde{\boldsymbol{d}}_h) \in V_h \times \widetilde{D}_h$ *solves* (3.23), *then* $(\boldsymbol{u}_h, P_{S_h}\boldsymbol{\epsilon}(\boldsymbol{u}_h) + \widetilde{\boldsymbol{d}}_h, \mathscr{C}(P_{S_h}\boldsymbol{\epsilon}(\boldsymbol{u}_h) + \widetilde{\boldsymbol{d}}_h))$ *solves* (3.4).

Following the same approach, the strain gap method proposed in [29] may readily be shown to be equivalent to the method of mixed enhanced strains.

We note in addition that the method of *enhanced assumed strains* [35] may be obtained by choosing the spaces in such a way that

$$\boldsymbol{\epsilon}(V_h) \subset S_h \subset D_h. \tag{3.24}$$

The projections in (3.23) then disappear, and we have the problem of finding $(\boldsymbol{u}_h, \widetilde{\boldsymbol{d}}_h) \in V_h \times \widetilde{D}_h$ that satisfy

$$\int_\Omega (\boldsymbol{\epsilon}(\boldsymbol{v}_h) + \widetilde{\boldsymbol{e}}_h) : \mathscr{C}(\boldsymbol{\epsilon}(\boldsymbol{u}_h) + \widetilde{\boldsymbol{d}}_h)\,\mathrm{d}x = \ell(\boldsymbol{v}_h), \quad (\boldsymbol{v}_h, \widetilde{\boldsymbol{e}}_h) \in V_h \times \widetilde{D}_h. \tag{3.25}$$

It is important to note, though, that (3.24) is not a necessary condition for obtaining an enhanced assumed strain formulation. This may be seen, for example, by returning to the basis proposed for $\widetilde{D}_h$ in [35], viz.

$$\widetilde{D}_h|_K = \operatorname{span}\begin{bmatrix} \xi & 0 & 0 & 0 \\ 0 & \eta & 0 & 0 \\ 0 & 0 & \xi & \eta \end{bmatrix} = B \oplus C. \tag{3.26}$$

Then the choice

$$D_h = \boldsymbol{\epsilon}(V_h) + B + C, \tag{3.27}$$

in which the sum is *not* direct, together with a choice for S_h such that $S_h \perp \widetilde{D}_h$, leads to the enhanced strain formulation (3.25). A suitable choice for S_h would be, for example,

$$S_h|_K = \mathrm{Id} + A; \tag{3.28}$$

this choice satisfies the conditions $S_h \perp \widetilde{D}_h$, but we do not have the inclusion $\boldsymbol{\epsilon}(V_h) \subset S_h$.

A posteriori recovery of the stress for the enhanced assumed strain method is discussed in [35].

It has been shown in [6] that the Hellinger–Reissner and enhanced assumed strain formulations are equivalent provided that

$$S_h \subset \mathscr{C}(\boldsymbol{\epsilon}(V_h) + \widetilde{D}_h), \tag{3.29}$$

while the equivalence of the Hellinger–Reissner and mixed enhanced strain formulations requires that [24]

$$\mathscr{C}^{-1}S_h \subset S_h \oplus \widetilde{D}_h. \tag{3.30}$$

One further class of such methods that warrants mention is that of the so-called B-bar methods [20]. By way of illustrating the relationship with other methods associated with the Hu–Washizu formulation, we focus on one special B-bar method, viz. the mean dilatation approach due to Nagtegaal et al. [25]. This method takes the form of finding $\boldsymbol{u}_h \in V_h$ that satisfies

$$2\mu(\operatorname{dev}\boldsymbol{\epsilon}(\boldsymbol{u}_h), \operatorname{dev}\boldsymbol{\epsilon}(\boldsymbol{v}_h))_0 + (\lambda+\mu)(\overline{\operatorname{div}}\,\boldsymbol{u}_h, \overline{\operatorname{div}}\,\boldsymbol{v}_h)_0 = \ell(\boldsymbol{v}_h) \quad \text{for all } \boldsymbol{v}_h \in V_h, \tag{3.31}$$

in which $\overline{\operatorname{div}}\,\boldsymbol{u}_h|_K = (\int_K \operatorname{div}\boldsymbol{v}_h\,\mathrm{d}x)/\mathrm{vol}(K)$.

The classical derivation of the mean dilatation method from the standard Hu–Washizu formulation (3.4) starts with the assumptions (see also [33])

$$D_h = S_h = \operatorname{dev}\boldsymbol{\epsilon}(V_h) \oplus \Delta_h \mathbf{1}, \tag{3.32}$$

where the space Δ_h is to be specified. Then writing

$$\boldsymbol{d}_h = \operatorname{dev}\boldsymbol{\epsilon}(\boldsymbol{u}_h) + \tfrac{1}{2}p_h\mathbf{1},$$

for some $p_h \in \Delta_h$ and noting that an arbitrary member $\boldsymbol{\tau}_h \in S_h$ can be expressed in the form

$$\boldsymbol{\tau}_h = 2\mu\boldsymbol{\epsilon}(\boldsymbol{v}_h) + q_h\mathbf{1},$$

where $q_h \in \Delta_h$, we obtain from $(3.4)_2$ and setting $\boldsymbol{v}_h = \mathbf{0}$ in the expression for $\boldsymbol{\tau}_h$,

$$p_h = P_{\Delta_h}\operatorname{div}\boldsymbol{u}_h. \tag{3.33}$$

Next, from $(3.4)_1$ we find that

$$\boldsymbol{\sigma}_h = 2\mu\operatorname{dev}\boldsymbol{\epsilon}(\boldsymbol{u}_h) + (\lambda+\mu)(P_{\Delta_h}\operatorname{div}\boldsymbol{u}_h)\mathbf{1}. \tag{3.34}$$

Finally, by substitution in $(3.4)_3$ we obtain

$$2\mu(\operatorname{dev}\boldsymbol{\epsilon}(\boldsymbol{u}_h), \operatorname{dev}\boldsymbol{\epsilon}(\boldsymbol{v}_h))_0 + (\lambda+\mu)(P_{\Delta_h}\operatorname{div}\boldsymbol{u}_h, P_{\Delta_h}\operatorname{div}\boldsymbol{v}_h)_0 = \ell(\boldsymbol{v}_h). \tag{3.35}$$

The choice $\Delta_h = P_0$, the space of piecewise constant functions, immediately yields (3.31).

One may arrive at the mean dilatation formulation by another route, in which we begin with the modified Hu–Washizu formulation (3.3), and with the assumptions

$$\boldsymbol{\epsilon}(V_h) \subset S_h = D_h. \tag{3.36}$$

Then returning to the displacement formulation (3.14) and using part (b) of Lemma 3.3 and the fact that $Q_h = P_{S_h}$, we easily find that

$$(Q_h\boldsymbol{\epsilon}(\boldsymbol{u}_h), \mathscr{C}_h Q_h\boldsymbol{\epsilon}(\boldsymbol{v}_h))_0 = 2\mu(\operatorname{dev}\boldsymbol{\epsilon}(\boldsymbol{u}_h), \operatorname{dev}\boldsymbol{\epsilon}(\boldsymbol{v}_h))_0 + (\theta-\mu)(P_{S_h^t}\boldsymbol{\epsilon}(\boldsymbol{u}_h), P_{S_h^t}\boldsymbol{\epsilon}(\boldsymbol{v}_h))_0 + (\lambda+\mu)(\overline{\operatorname{div}}\,\boldsymbol{u}_h, \overline{\operatorname{div}}\,\boldsymbol{v}_h)_0, \tag{3.37}$$

where θ is given in Lemma 3.3. It follows that the modified Hu–Washizu formulation with α chosen such that $\theta = \mu$, leads to the mean dilatation formulation. This is so in particular for Case IV in Table 1, which satisfies (3.36).

More broadly, Simo and Hughes [32] have shown that any assumed strain method is variationally consistent with the Hu–Washizu formulation, and that provided that the displacement-based approach is consistent with this framework, the space of discrete stresses can be defined in such a way that the stresses can be recovered in a variationally consistent manner.

Remark. It will be seen in Section 4 that (3.6) is essential for stability. If this is absent then the resulting standard and modified Hu–Washizu formulations are distinct, and for an appropriate choice of bases the modified formulation with $\alpha \neq 1$ can be shown to lead to stable and convergent results while the standard formulation locks. Fig. 1 summarises the inter-relationships between the various formulations, as set out above. It is seen above and in Fig. 1 that the modified and

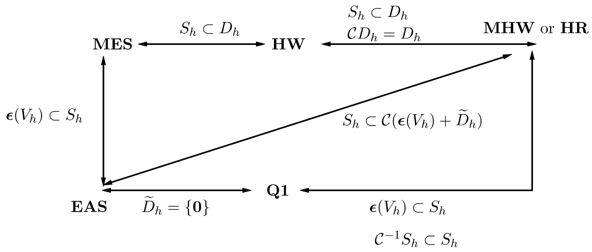

Fig. 1. Conditions for equivalence between the Hellinger–Reissner (HR), classical (HW) and modified Hu–Washizu (MHW), mixed enhanced strain (MES), enhanced assumed strain (EAS), and the classical Q_1 formulation.

classical Hu–Washizu problems are equivalent under the same set of assumptions that guarantee equivalence between the Hu–Washizu and Hellinger–Reissner problems. The question therefore arises as to whether it might not be more convenient to work with the two-field Hellinger–Reissner formulation than with the three-field formulations. Despite the fact that the Hu–Washizu formulation involves three variables, it carries certain advantages over the Hellinger–Reissner problem. Most particularly, the displacement is one of the primary variables in the Hu–Washizu formulations, so that static condensation leads naturally to a displacement-based formulation, which is advantageous in many respects. For example, in problems of elastoplasticity the use of a stress-based Hellinger–Reissner formulation does not permit the conventional highly effective return mapping algorithm to be applied at integration points; instead the computations must be carried out at element level [34]. We therefore take as basic the Hu–Washizu formulation.

Case II in Table 1 corresponds to the method of mixed enhanced strains, while Case V is equivalent to an enhanced assumed strain method. This is not the same, however, as the enhanced assumed method obtained from (3.26) or, equivalently, the classical Wilson incompatible modes element [39,35]. For this case we have D_h given by (3.27), and $\mathscr{C}D_h = D_h$. With the choice $S_h = \mathrm{Id} + A$ we have $S_h \subset \mathscr{C}D_h$ and the conditions for equivalence with a Hellinger–Reissner and classical Hu–Washizu formulation are satisfied.

We also note that Cases II, III and V satisfy the assumptions for equivalence between the classical and modified Hu–Washizu formulations, but that Cases I and IV do not satisfy the second of these.

Case IV with the choice $\alpha = 1$, that is, the classical Hu–Washizu formulation, is degenerate in that it leads to the standard Q_1-approach. This may be read off from Table 1, in that for Case IV we have $\epsilon(V_h) \subset S_h = D_h$, which implies also that $\widetilde{D}_h = \{\mathbf{0}\}$, so that the equivalence between the classical Hu–Washizu and Q_1 formulations is arrived at via the route $\mathrm{HW} \equiv \mathrm{MES} \equiv \mathrm{EAS} \equiv \mathrm{Q1}$.

Furthermore, as is shown in [23], Case IV with $\alpha = -\frac{\mu}{\lambda}\frac{3\lambda+2\mu}{2\lambda+\mu}$ is equivalent to the penalized $Q_1 - P_0$ problem of finding $\boldsymbol{u}_h \in V_h$ and $p_h \in \widetilde{M}_h$ such that

$$
\begin{aligned}
&2\mu(\boldsymbol{\epsilon}(\boldsymbol{u}_h), \boldsymbol{\epsilon}(\boldsymbol{v}_h))_0 + (p_h, \operatorname{div} \boldsymbol{v}_h)_0 = \ell(\boldsymbol{v}_h), \quad \boldsymbol{v}_h \in V_h,\\
&(\operatorname{div} \boldsymbol{u}_h, q_h)_0 - \lambda^{-1}(p_h, q_h)_0 = 0, \quad q_h \in \widetilde{M}_h.
\end{aligned}
$$

4. Numerical results

In this section, we present a selection of numerical simulations related to the formulations and bases discussed in Section 3. All examples are considered under conditions of plane strain, and are based on four-noded quadrilateral elements with standard bilinear interpolation of the displacement field.

We are interested in particular in demonstrating computationally the stability of those schemes that satisfy the conditions for well-posedness set out in Section 3, and also illustrating pathological behavior such as locking, in those cases in which the conditions for well-posedness are not met. A further phenomenon concerns the spurious modes that are associated with the $Q_1 - P_0$ element, and which according to the theory presented earlier should be anticipated.

Unless otherwise stated, all examples are presented for the case of near-incompressibility, and correspond to a value of Poisson's ratio close to 0.5.

The formulations referred to in the examples that follow are:

- Q1 The standard displacement formulation using the four-noded quadrilateral.
- HWI The standard Hu–Washizu formulation with the bases corresponding to Case I.
- MHWI The modified Hu–Washizu formulation with $\alpha = 0$, and with the bases corresponding to Case I. This is *not* equivalent to HWI.
- MES The method of mixed enhanced strains, which is equivalent to Case II, and using the bases proposed in [21]. This formulation is independent of α.

HWIII The Hu–Washizu formulation with the bases corresponding to Case III. This formulation is independent of α.
HWIV The standard Hu–Washizu formulation with the bases corresponding to Case IV.
MHWIV The modified Hu–Washizu formulation with $\alpha = 0$, and with bases corresponding to Case IV. This is *not* equivalent to HWIV.
EAS The method of enhanced assumed strains, which is equivalent to the α-independent Hu–Washizu formulation using the bases in Case V in Table 1, and which corresponds to one of the sets of bases proposed in [35].

Example 1 (*Cook's membrane problem*). This benchmark problem, shown in Fig. 2, refers to a tapered panel clamped on one side and subjected to a shearing load at the free end, resulting in deformation that is dominated by a bending response. The material properties are taken to be $E = 250$ and $\nu = 0.4999$, and simulations are carried out for progressive uniform refinements of the mesh.

In Fig. 2 the results obtained from various formulations are compared. The well-known locking response of Q1 is observed, as is the corresponding poor responses of HWI and HWIV, which cases are not covered by the well-posedness theory. On the other hand, MHWI shows no such pathologies, and performs well, while HWIII, which corresponds to constant direct stresses and strains, exhibits a response that is most accurate. The EAS and MES formulations produce characteristically good results.

Example 2 (*Cantilever beam*). We consider a beam of unit thickness, subjected to a couple at one end, as shown in Fig. 3. Along the edge $x = 0$, the horizontal displacement and vertical surface traction are zero. At the point $(0,0)$, the vertical displacement is also zero. The exact solution is given by

$$u(x,y) = \frac{2f(1-\nu^2)}{El}x\left(\frac{l}{2}-y\right) \quad \text{and} \quad v(x,y) = \frac{f(1-\nu^2)}{El}\left[x^2 + \frac{\nu}{1-\nu}y(y-l)\right].$$

We set $L = 10$, $l = 2$, $E = 1500$, $\nu = 0.4999$, and $f = 3000$.

Fig. 3 shows the behavior of the displacement error

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_1^2 = \|\boldsymbol{u} - \boldsymbol{u}_h\|_0^2 + \|\nabla\boldsymbol{u} - \nabla\boldsymbol{u}_h\|_0^2$$

with mesh refinement, for the case of a series of regular refinements. Meshes of 2×1, 4×2, 8×4 and 16×8 rectangular and distorted elements are considered.

In Fig. 3 it is apparent the convergence rate of that both the standard formulation and the Hu–Washizu formulation HW1 are very poor and almost identical, while other formulations exhibit a rate of convergence close to linear.

Turning next to the distorted meshes, as illustrated in Fig. 4, the displacement error exhibits behavior almost identical to that for the meshes of rectangular elements.

Example 3 (*Thick-walled cylinder*). This problem has been considered in [21,35], and consists of a thick-walled cylinder subjected to an internal pressure p. Plane strain is assumed in the axial direction, with radial displacement $u(r)$ given by

$$u(r) = \frac{(1+\nu)pR_i^2}{E(R_o^2 - R_i^2)}[R_o^2/r + (1-2\nu)r],$$

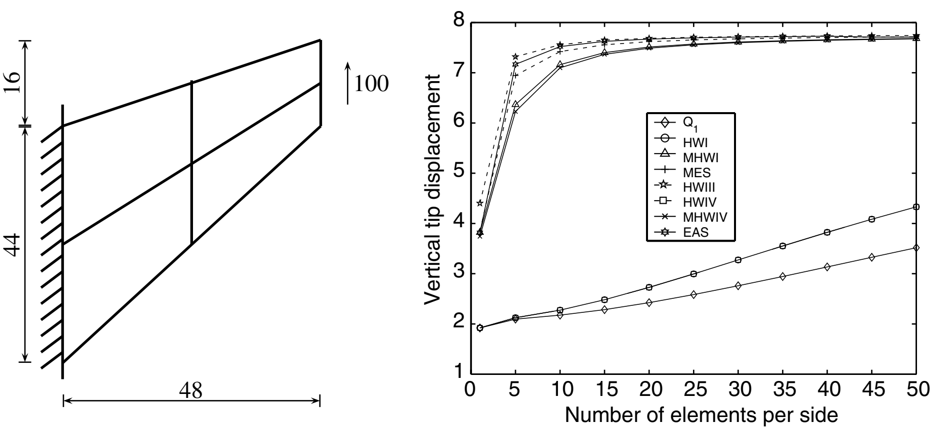

Fig. 2. Cook's membrane problem.

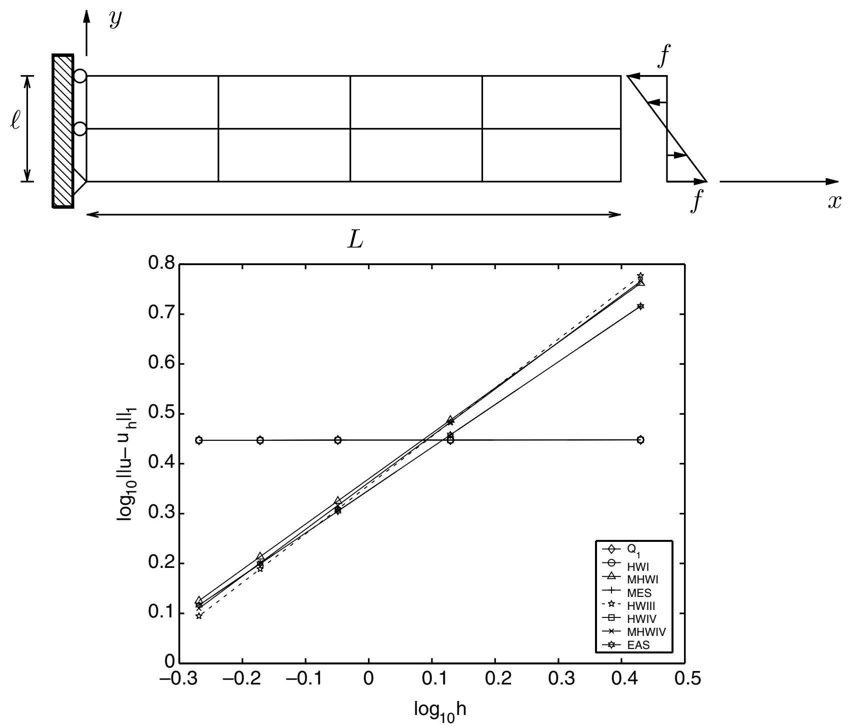

Fig. 3. Behavior of displacement error, measured in the H^1 norm, with mesh size, using regular refinements of a rectangular mesh.

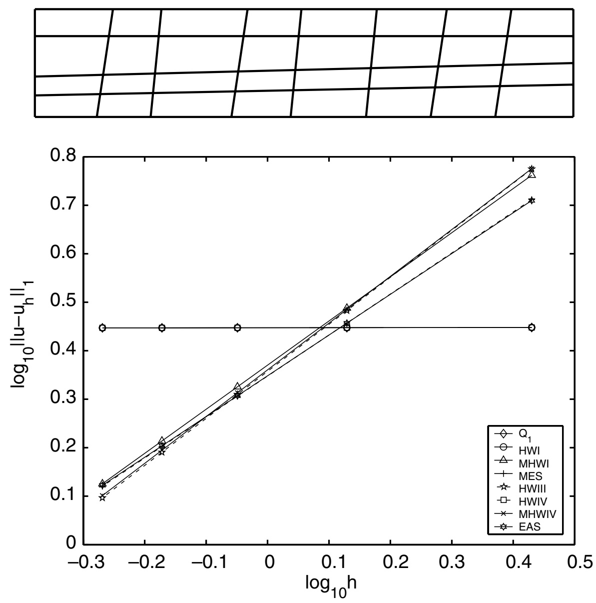

Fig. 4. Behavior of displacement error, measured in the H^1 norm, with mesh size, using regular refinements of the distorted mesh shown.

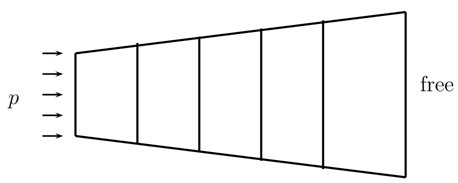

Fig. 5. Finite element mesh for thick-walled cylinder.

where R_i and R_o are respectively the inner and outer radii. The problem is analyzed as a two-dimensional problem, with a sector of 10 degrees and five trapezoidal elements as shown in Fig. 5. A dimensionless pressure $p/E = 10^{-3}$ is used, with $R_i = 3$ and $R_o = 9$.

Table 2
Normalized radial displacement at the inner radius, for the thick cylinder subjected to internal pressure

v	Q1	HWI	MHWI	HWIII	EAS	HWIV	MHWIV
0.49	0.8218	0.8290	0.9542	0.9693	0.9659	0.8218	0.9464
0.499	0.3291	0.3516	0.9524	0.9692	0.9656	0.3505	0.9473
0.4999	0.0489	0.0520	0.9522	0.9692	0.9656	0.0520	0.9474
0.49999	0.0051	0.0055	0.9522	0.9692	0.9656	0.0050	0.9474

Radial displacements at the inner radius are compared with the exact solution for the various formulations, and for different values of Poisson's ratio. Table 2 gives the values of u_h/u at the inner radius. These again show the locking behavior of Q1, HWI, and HWIV, while the other formulations exhibit good performance. Results for MES are not shown separately since these are identical to those obtained for EAS.

Example 4 (*Driven cavity*). We consider the problem of a unit square subjected to a unit horizontal displacement along the upper boundary. The material properties used are Young's modulus $E = 0.1$, Poisson's ratio $v = 0.4999$, and $\lambda/\mu = 10^7$. The aim here is demonstrate numerically the existence of the checkerboard modes predicted by the theory of Section 3.

The hydrostatic pressure p is computed from $\mathrm{tr}\,\boldsymbol{\sigma}_h = \sigma_{h1} + \sigma_{h2}$, for uniform meshes of square elements with 20 and 40 elements per side. The pressure distribution along the line $y = 0.22$ is shown for Case I with the standard and modified Hu–Washizu formulations, in Fig. 6. Case HWI exhibits a poor response which does not even reproduce the sinusoidal-like pressure characteristic of this problem. For Case MHWI the checkerboard mode is easily seen.

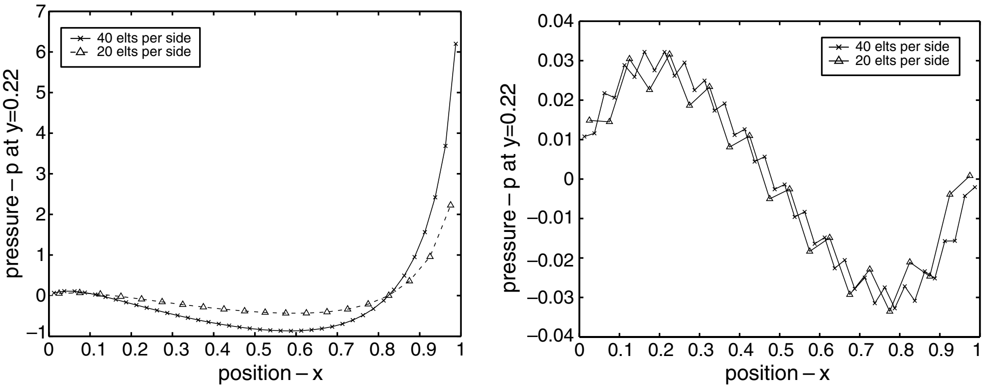

Fig. 6. Driven cavity flow: pressure distribution $y = 0.22$ for problems HWI (left) and MHWI (right).

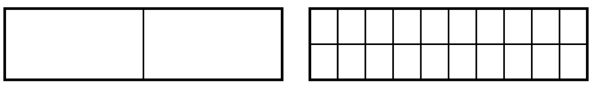

Fig. 7. The Cantilever beam with coarse and fine meshes.

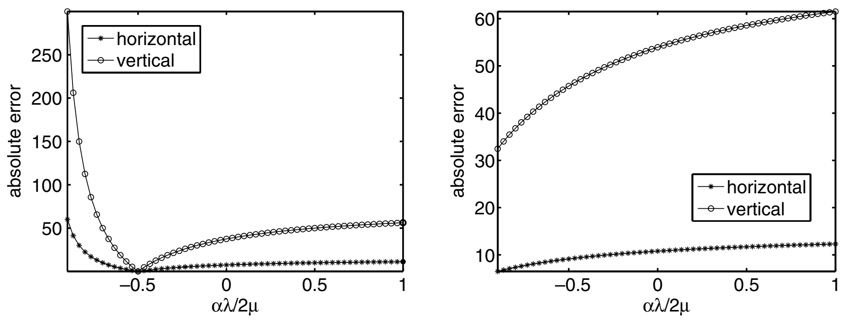

Fig. 8. Tip displacement error versus $\frac{\alpha\lambda}{2\mu} \in (-1, 1]$ for the coarse mesh, Cases MHWI (left) and MHWIV (right).

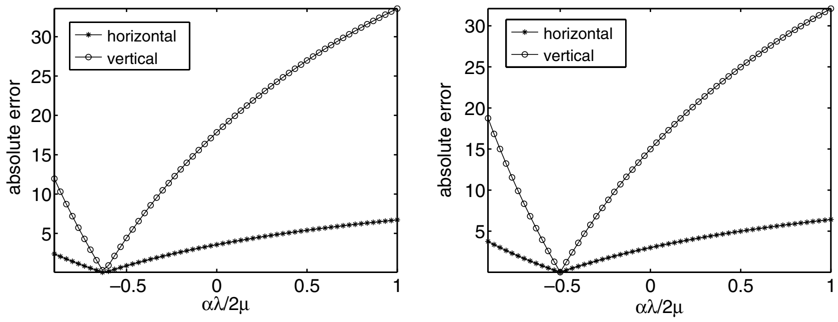

Fig. 9. Tip displacement error versus $\frac{\alpha\lambda}{2\mu} \in (-1, 1]$ for the fine mesh, Cases MHWI (left) and MHWIV (right).

Example 5 (*Cantilever beam*: *behavior with variation in* α). In this final example, we elaborate on the dependence of the solution on the parameter α by considering again Example 2, with the two meshes illustrated in Fig. 7. The variation with $\alpha\lambda/2\mu$ of the absolute error of the vertical and horizontal tip displacements at the top right-hand corner (that is, at the point $(10, 2)$ in Fig. 3), is shown in Fig. 8 for Cases MHWI and MHWIV, for the 2×1 mesh, and in Fig. 9 for the 10×2 mesh. To avoid the singularity at $\alpha = -\mu/\lambda$, we use the projection-based formulation derived in [23, Lemma 5.5]. For $\alpha = -2\mu/\lambda$, the Case MHWIV reduces to HWIII. As can be seen from Figs. 8 and 9, with a coarse mesh the locking effect increases for larger $\alpha\lambda$, and there is a singularity at $\alpha = -2\mu/\lambda$ for Case MHWI. We recall that $\alpha = 1$ corresponds to the standard Hu–Washizu formulation.

5. Concluding remarks

This work extends and elaborates on an analysis of the Hu–Washizu problem presented in [23]. In that work a one-parameter family of Hu–Washizu formulations has been presented and analysed. Here the relationship between the generalized and classical formulations is clarified through recourse to the underlying saddlepoint problems.

The equivalence between the Hu–Washizu and various enhanced and other related formulations has been explored, resulting in a unified picture that incorporates both classical equivalence results and newer results flowing from the current work.

The implications of the general theory, particularly the results on well-posedness, have been explored in the context of five different sets of bases, some of which are well-known. Locking behavior anticipated by the theory is verified numerically; likewise, the checkerboard modes present in the spherical part of the stress are demonstrated numerically. Results for those bases satisfying the conditions for well-posedness reflect the high accuracy anticipated, and are in line with well-known results for the enhanced assumed strain and related methods.

Problems of near-incompressibility are of considerable relevance in the nonlinear regime. While many of the theories developed in the linear context have been applied, often with much success, to problems involving nonlinear geometric and material behavior, the underpinning theory is still in an incomplete state. The work [7] represents an encouraging step in the direction of the development of such a theory, but much remains to be done, for example in analysing the Hu–Washizu formulation in its nonlinear context.

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