On the mathematical formulations of a model of strain gradient plasticity

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1 Introduction

The inability of classical or conventional plasticity to model material behaviour at nano/meso scale levels has led to the development of new theories of plasticity in which size dependence is captured through the inclusion of either higher order derivatives of internal variables or higher order stresses in the yield criterion. For instance, in the pioneering works of Aifantis [1], Mühlhaus and Aifantis [14] the classical yield condition is augmented by a term involving the Laplacian of the effective plastic strain. Many other approaches have been presented in the literature. We can mention among all the papers [6, 9, 10].
While most of the new theories are based on the classical assumption that the direction of plastic flow is governed by the deviatoric part of the Cauchy stress, Gudmunson [9] and then Gurtin and Anand [10] proposed a different approach by introducing a microstress T^p which is power conjugate to the plastic strain rate $\dot{E}^p$ and a third order micropolar stress $\mathbb{K}^p$ which is power conjugate to the gradient of the plastic strain rate $\nabla\dot{E}^p$. These two tensors are related to the deviatoric part of the Cauchy tensor T through a microscopical force balance (2) which supplements the classical equilibrium (1).
The purpose of this paper is to study the variational formulations of the model of strain gradient plasticity for isotropic, plastically irrotational material, under small deformation, by Gurtin and Anand [10]. In [15], using the convex analytical setting of classical plasticity developed in [11], the flow law is written in its primal form and hence, combining both macroscopical and microscopical force balances together with the flow law, the model is formulated as a variational inequality whose well-posedness is studied in the case of hardening behaviour. We present in the first part of this paper a brief account of the formulation in [15].
As discussed in [15, Section 4], it is quite impossible to accommodate soften-

ing behaviour in the formulation as variational inequality. On the other hand, softening behaviour is difficult to handle even for classical plasticity when formulated as variational inequality. A different formulation which accounts for plasticity with softening behaviour has been proposed recently by by Dal Maso, De Simone, Mora and Morini in [3, 4] following the energetic approach for rate-independent processes developed by Mielke in [12, 13].

In a recent paper by Giaccomini and Lussardi [7], an existence result for the Gurtin-Anand's model within the Mielke's energetic-approach is obtained and it has also been proved that the model converges in a suitable sense to the formulation of classical perfect plasticity in [2] whenever the energetic and dissipative length scales L in (7) and ℓ in (12) respectively go to zero.

Inspired by the results in the papers [3, 4, 7], we present in the second part of this paper (see Section 3) an energetic formulation for the quasi-static evolution of the Gurtin-Anand's model of strain gradient plasticity with softening behaviour. We refer the reader to the forthcoming paper [5] for the details of our analysis.

2 The governing equations for the problem

Let $\Omega \subset \mathbb{R}^3$ be a bounded connected open set which Lipschitz boundary $\partial\Omega = \Gamma_D \cup \Gamma_N$ which is occupied by an elastoplastic body in its undeformed configuration. The body is assumed to undergo infinitesimal deformations.

The model proposed by Gurtin and Anand [10] for small-deformation gradient plasticity is characterized by the inclusion of a second order tensor T^p, the microstress and a (third order tensor) polar microstress $\mathbb{K}^p$ which are conjugate respectively to the plastic strain rate $\dot{E}^p$ and gradient of plastic strain rate $\nabla\dot{E}^p$. The set of equations and boundary conditions below are obtained in [10] as consequences of the balance of internal and external power expenditures. In addition, the theory assumes zero plastic rotation. We confine attention to the case of time-independent plastic behaviour, though the theory in [10] is valid for viscoplastic materials.

Balance equations and boundary conditions. The conventional macroscopic force balance leads to the equation of equilibrium

$$\operatorname{div} T + f = 0 \tag{1}$$

in which T is the symmetric Cauchy stress and f is the body force. Due to the microstresses, the equilibrium equation above is augmented by the microforce balance

$$T^D = T^p - \mathrm{div}\mathbb{K}^p\ . \tag{2}$$

Here and henceforth $T^D_{ij} = T_{ij} - \frac{1}{3}\delta_{ij}\mathrm{tr}T$ denotes the deviatoric part of the second-order tensor T. The divergence of the third-order polar microstress is the second-order tensor with components

$$(\mathrm{div}\mathbb{K})_{ij} = K_{ijk,k}\ , \tag{3}$$

and in which a subscript following a comma denotes partial differentiation with respect to that spatial component.
The macroscopic boundary conditions are given by

$$u = 0 \quad \text{on } \Gamma_D, \qquad t := Tn = \bar{t} \quad \text{on } \Gamma_N \ . \tag{4}$$

That is, we assume homogeneous Dirichlet boundary conditions on a part Γ_D of the boundary $\partial\Omega$, and a prescribed surface traction on the complement $\Gamma_N := \partial\Omega \backslash \Gamma_D$. The outward unit normal to $\partial\Omega$ is denoted by ν. It is assumed that $\Gamma_D \neq \varnothing$. It is necessary also to prescribe microscopic boundary conditions on $\partial\Omega$; these take the form

$$E^p = 0 \quad \text{on } \Gamma_H, \qquad \mathbb{K}^p n = 0 \quad \text{on } \Gamma_F \ . \tag{5}$$

The subscripts 'H' and 'F' denote 'hard' and 'free' parts of the boundary respectively (see [10], Section 8), and Γ_H and Γ_F are complementary subsets of $\partial\Omega$. [3] The microscopic boundary conditions in (5) implies $\int_{\partial\Omega} \mathbb{K}^p n : \dot{E}^p dV = 0$ which is called in [10] *null microscopic power expenditure* on the boundary $\partial\Omega$.
Constitutive theory. The constitutive equations are obtained from a free energy imbalance together with a flow law that characterizes plastic behaviour. The total strain E is additively decomposed into elastic and plastic components E^e and E^p, so that

$$E = E^e + E^p \tag{6}$$

with the plastic strain E^p satisfying $\mathrm{tr}E^p = 0$.
The relation strain-displacement is given by $Eu = (\nabla u + \nabla u^T)/2$.
The Burgers tensor G is a tensor of second order, equal to the curl of the plastic strain: that is, $G = \mathrm{curl}\, E^p$ or $G_{ij} = \varepsilon_{ijk} E^p_{ik,j}$, where ε_{ijk} is the permutation symbol.
In this paper (see also [15]) we modify the model presented in [10] in order to account for linear isotropic hardening. Therefore following the approach in [11] for classical plasticity, we consider here a free energy of the form

$$\psi(E^e, E^p, \gamma) := \underbrace{\frac{1}{2} E^e : \mathcal{C} E^e}_{\text{elastic energy}} + \underbrace{\frac{1}{2}\mu L^2 |\mathrm{Curl}\, E^p|^2}_{\text{defect energy}} + \underbrace{\frac{1}{2} k |\gamma|^2}_{\text{hardening energy}} \tag{7}$$

where $\mathcal{C}$ is the elasticity tensor which for homogeneous isotropic media is defined by $\mathcal{C}E = \lambda(\mathrm{tr}E)I + \mu(E + E^T)$ for any second-order tensor E, λ and μ called the Lame moduli, L is an energetic length scale and k is a positive constant.
The local free-energy imbalance states that

$$\dot{\psi} - T : \dot{E}^e - T^p : \dot{E}^p - \mathbb{K}^p : \nabla \dot{E}^p \leq 0 \ . \tag{8}$$

[3] See Section 8 of G&A for situation in which $\mathbb{K}^p_{diss} = 0$.

Expansion of the first term and substitution of (7) gives the elastic relation

$$T = \mathcal{C}E^e \ . \tag{9}$$

Now, if we set

$$P_{jqp} = \varepsilon_{ipq}\frac{\partial\hat{\psi}}{\partial G_{ij}}, \quad \mathbb{K}^p_{\text{dis}} = \mathbb{K}^p - \mathbb{P} \quad \text{and} \quad g = -\frac{\partial\hat{\psi}}{\partial\gamma} = k\gamma \tag{10}$$

then the dissipation inequality becomes

$$T^p : \dot{E}^p + \mathbb{K}^p_{\text{dis}} : \nabla\dot{E}^p + g\dot{\gamma} \geq 0 \ . \tag{11}$$

The flow law. Set $\varSigma^p = (T^p, \mathbb{K}^p_{dis}, g)$. We have the yield criterion

$$\phi(\varSigma^p) = \sqrt{|T^p|^2 + \ell^{-2}|\mathbb{K}^p_{\text{dis}}|^2} + g - S_Y \leq 0 \tag{12}$$

where ℓ is a dissipative length scale.
So the set of admissible generalized stresses is

$$\mathcal{K} := \left\{\varSigma^p\colon \sqrt{|T^p|^2 + \ell^{-2}|\mathbb{K}^p_{\text{dis}}|^2} + g - S_Y \leq 0\right\} \ . \tag{13}$$

If $\varGamma^p = (E^p, \nabla E^p, \gamma)$ then the maximum dissipation principle gives the normality law

$$\dot{\varGamma}^p \in N_{\mathcal{K}}(\varSigma^p) \tag{14}$$

where $N_{\mathcal{K}}(\varSigma^p)$ denotes the normal cone to $\mathcal{K}$ at $\varSigma^p$, which is the set of generalised strains $\dot{\varGamma}^p$ that satisfy $(\overline{\varSigma} - \varSigma^p) : \dot{\varGamma}^p \leq 0 \quad$ for all $\overline{\varSigma} \in \mathcal{K}$.
Since $\partial\mathcal{K}$ is smooth, this gives $\dot{\varGamma}^p = \lambda\dfrac{\partial\phi}{\partial\varSigma^p}$. That is, $\dot{E}^p = \lambda\dfrac{T^p}{S_Y - g}$, $\nabla\dot{E}^p = \lambda\ell^{-2}\dfrac{\mathbb{K}^p_{\text{dis}}}{S_Y - g}$ and $\dot{\gamma} = \lambda$.
Introduce

$$d^p(\dot{E}^p) := \sqrt{|\dot{E}^p|^2 + \ell^2|\nabla\dot{E}^p|^2} = \dot{\gamma} = \lambda$$

Using convex analysis we find that

$$\dot{\varGamma}^p \in N_{\mathcal{K}}(\varSigma^p) \quad \Leftrightarrow \quad \varSigma^p \in \partial\mathcal{D}(\dot{\varGamma}^p) \tag{15}$$

where $\mathcal{D}$ is the dissipation function which in this case is defined by

$$\mathcal{D}(M) := \begin{cases} S_Y\, d^p(Q) & \text{if } d^p(Q) \leq \xi, \\ \infty & \text{otherwise}. \end{cases} \quad \forall M = (Q, \nabla Q, \xi) \tag{16}$$

and $\partial\mathcal{D}(\dot{\varGamma}^p)$ denotes the subdifferential of $\mathcal{D}$ evaluated at $\dot{\varGamma}^p$. That is,

$$\varSigma^p \in \partial\mathcal{D}(\dot{\varGamma}^p) \quad \Leftrightarrow \quad \mathcal{D}(M) \geq D(\dot{\varGamma}^p) + \varSigma^p : (M - \dot{\varGamma}^p) \quad \text{for any } M. \tag{17}$$

The formulation as variational inequality. Assume for instance that $\Gamma_{\mathrm{N}} = \emptyset$ and that $\overline{u} = 0$. Taking the inner product of the equilibrium equation (1) with $v - \dot{u}$ where v is an arbitrary function v which satisfies the Dirichlet boundary condition $(4)_1$, integrating over Ω, and then integrating by parts, also substituting (9) for T, we obtain

$$\int_\Omega \mathcal{C}(Eu - E^p) : (Ev - E\dot{u})\ dx = \int_\Omega f \cdot (v - \dot{u})\ dx\ . \tag{18}$$

Next, expanding (17) and integrating over Ω we have

$$\begin{aligned}\int_\Omega \mathcal{D}(Q, \nabla Q, \mu)\ dx &\geq \int_\Omega \mathcal{D}(\dot{E}^p, \nabla \dot{E}^p, \dot{\gamma})\ dx + \int_\Omega T^p : (Q - \dot{E}^p)\ dx \\ &\qquad + \int_\Omega \mathbb{K}^p_{\mathrm{dis}} : \nabla(Q - \dot{E}^p)\ dx + \int_\Omega g(\mu - \dot{\gamma})\ dx \\ &\geq \int_\Omega \mathcal{D}(\dot{E}^p, \nabla \dot{E}^p, \dot{\gamma})\ dx + \int_\Omega T : (Q - \dot{E}^p)\ dx \\ &\qquad - \int_\Omega \mathbb{P} : \mathrm{curl}\,(Q - \dot{E}^p)\ dx + \int_\Omega g(\mu - \dot{\gamma})\ dx \end{aligned} \tag{19}$$

where the last inequality follows from the microscopic force balance (2), the definition of $\mathbb{K}^p_{\mathrm{dis}}$ and $\mathbb{P}$ in (10) respectively and the microscopic boundary conditions (5) satisfied by Q and E^p.

Now following the formulation developed in [11] for classical plasticity, we introduce the spaces of functions

$$\mathcal{V} = [H^1_0(\Omega)]^3,\ \mathcal{Q} = \left\{ \begin{array}{l} Q = (Q_{ij}) \in [H^1(\Omega)]^{3\times 3} :\ Q_{ji} = Q_{ij} \\ \mathrm{tr}\, Q = 0 \text{ a.e. in } \Omega,\ Q = 0 \text{ on } \Gamma_H \end{array} \right\},\ \mathcal{M} = L^2(\Omega)$$

These are Hilbert spaces when endowed respectively with the norms

$$\|v\|_{\mathcal{V}} = \|\nabla v\|_0\ ,\ \ \|Q\|_{\mathcal{Q}} = \left(\|Q\|_0^2 + \|\nabla Q\|_0^2\right)^{1/2} \text{ and } \|\xi\|_{\mathcal{M}} = \|\xi\|_0\ . \tag{20}$$

For convenience the L^2-norm, whether for scalar-, vector- or tensor-valued functions, is denoted by $\|\cdot\|_0$. Now, set

$$\mathcal{Z} = \mathcal{V}\times\mathcal{Q}\times\mathcal{M} \quad \text{and} \quad \mathcal{W} = \left\{ (v, Q, \xi) \in \mathcal{Z} : \sqrt{|Q|^2 + \ell^2|\nabla Q|^2} \leq \xi \text{ a.e. in } \Omega \right\}$$

Then $\mathcal{Z}$ is a Hilbert space with norm $\|z\|_{\mathcal{Z}} = \left(\|v\|^2_{\mathcal{V}} + \|Q\|^2_{\mathcal{Q}} + \|\xi\|^2_{\mathcal{M}}\right)^{1/2}$ and $\mathcal{W}$ is a non-empty closed convex cone in $\mathcal{Z}$.

Now adding up (18) and (19), we find that $w = (u, E^p, \gamma)$ solves the variational inequality

$$a(w(t), z - \dot{w}(t)) + j(z) - j(\dot{w}(t)) - \ell(z - \dot{w}(t)) \geq 0 \qquad \forall z = (v, Q, \xi) \in \mathcal{W}$$

where

$$a(w,z) = \int_\Omega \mathcal{C}(Eu - E^p) : (Ev - Q)\, dx + k\int_\Omega \gamma\xi dx + \alpha L^2 \int_\Omega \mathrm{curl} E^p : \mathrm{curl} Q\, dx \tag{21}$$

$$j(z) := \begin{cases} \int_\Omega \mathcal{D}(Q, \nabla Q, \xi)\ dx & \text{if } z \in \mathcal{W}, \\ +\infty & \text{otherwise,} \end{cases} \quad \text{and} \quad \ell(z) := \int_\Omega f \cdot v\ dx\,. \tag{22}$$

The well-posedness of the Gurtin-Anand's model formulated as variational inequality is obtained in [15] applying [11, Theorem 7.3].

3 Energetic formulation for softening behaviour

The analysis in [15, Section 4] has shown the impossibility for the Gurtin-Anand's model to accommodate softening behaviour (corresponding in this case to $k < 0$) in the formulation as variational inequality. In fact, we loose the coercivity of the bilinear form a on $\mathcal{Z}$ or $\mathcal{W}$ when $k < 0$.
We present in this section an energetic formulation in the spirit of [12, 13, 3, 4, 7], which accommodates softening behaviour.

Prescribed boundary displacement. To simplify the problem, we assume that there are no volume forces nor traction prescribed. So, the evolution is only governed by a prescribed boundary displacement on $\partial\Omega$ which is given by the trace of a function $w : [0,\infty) \times \Omega \to \mathbb{R}^3$ which is locally absolutely continuous in time with values in $H^1(\Omega, \mathbb{R}^3)$. For instance, $w \in H^1_{loc}(0,\infty; H^1(\Omega, \mathbb{R}^3))$.

Energies and potentials. The energetic formulation of the quasi-static evolution of Gurtin-Anand's model with softening behaviour involves

$$\mathcal{Q}_1(E^e) := \frac{1}{2}\int_\Omega E^e : \mathcal{C}E^e dx, \quad \mathcal{Q}_2(E^p) = \frac{1}{2}\mu L^2 \int_\Omega |\mathrm{curl} E^p|^2 dx, \tag{23}$$

$$\mathcal{V}(\xi) := \int_\Omega V(\xi(x)) dx \tag{24}$$

where the potential $V : \mathbb{R} \to \mathbb{R}$ is the softening potential which is assumed here to be a function of class C^2 such that there exists $M > 0$ such that

$$-M \le V''(\xi) \le 0 \quad \forall \xi, \quad -\frac{1}{2}S_{\mathrm{Y}} < V'(+\infty) \le V'(-\infty) < \frac{1}{2}S_{\mathrm{Y}} \tag{25}$$

where S_{Y} is the yield constant.

Remark 1. A typical potential V is given by $V(\xi) = -c(1 + |\xi|^2)^{1/2}$ with $0 < c < \frac{1}{2}S_{\mathrm{Y}}$.

We find that the dissipation function corresponding to softening behaviour is given by

$$\mathcal{D}(M) = S_{\rm Y} \max\Big(\sqrt{|Q|^2 + \ell^2|\nabla Q|^2}, \xi\Big) \quad \forall M = (Q, \nabla Q, \xi)\,. \tag{26}$$

Notice that in this case the functional $\mathcal{H}(Q,\xi) := \int_\Omega \mathcal{D}(Q,\nabla Q,\xi)dx$ is defined in $W^{1,1}(\Omega, M^{3\times3}) \times L^1(\Omega)$. However, $\mathcal{H}$ will be involved in an analysis (the incremental problem) which employs the direct method of calculus of variations. Therefore, the process of relaxation with respect to weak convergence (see [8]) in the space $BV(\Omega, M^{3\times3}) \times L^1(\Omega)$ will give for any $(Q,\xi) \in BV(\Omega, M^{3\times3}) \times L^1(\Omega)$

$$\mathcal{H}(Q,\xi) = S_{\rm Y}\int_\Omega \max\Big(\sqrt{|Q(x)|^2 + \ell^2|\nabla Q(x)|^2}, \xi(x)\Big)dx + S_{\rm Y} l |D^s Q|(\Omega) \tag{27}$$

where ∇Q is the so-called approximate gradient of Q and D^sQ is the singular part of the measure DQ with respect to the Lebesgue measure $\mathcal{L}^3$.

Remark 2. Notice that $\mathcal{H}(Q_1+Q_2, \xi_1+\xi_2) \le \mathcal{H}(Q_1,\xi_1) + \mathcal{H}(Q_2,\xi_2)$ and there exists $m > 0$ such that for any (Q_1,ξ_1), $(Q_2,\xi_2) \in BV(\Omega, M^{3\times3}) \times L^1(\Omega)$

$$\mathcal{H}(Q_1-Q_2, \xi_1-\xi_2) + \mathcal{V}(\xi_1) - \mathcal{V}(\xi_2) \ge m\,[||Q_1-Q_2||_{BV} + ||\xi_1-\xi_2||_{L^1}]\,. \tag{28}$$

The dissipation of a function $t \to (Q(t),\xi(t))$ on an interval $[a,b] \subset [0,\infty)$

$$\mathcal{D}_{\mathcal{H}}(Q,\xi;a,b) = \sup\left\{\sum_{j=1}^k \mathcal{H}(Q(t_j)-Q(t_{j-1}), \xi(t_j)-\xi(t_{j-1})),\right\} \tag{29}$$

where the supremum is taken over all partitions $t_0 = a < t_1 < \cdots < t_{k-1} < t_k = b$ of $[a,b]$.
When $t \to (Q(t),\xi(t))$ is smooth then $\mathcal{D}_{\mathcal{H}}(Q,\xi;a,b) = \int_a^b \mathcal{H}(\dot{Q}(t),\dot{\xi}(t))dt\,.$

Admissible configurations. Let $w \in H^1(\Omega, \mathbb{R}^3)$ be a boundary data prescribed on $\partial\Omega$. An admissible configuration relative to w is given by (u, E^e, E^p, ξ) such that

$$u \in W^{1,3/2}(\Omega,\mathbb{R}^3),\ E^e \in L^2(\Omega, M^{3\times3}_{sym}),\ E^p \in BV(\Omega, M^{3\times3}_{sym}),\ \xi \in H^1(\Omega) \tag{30}$$

$$u = w \text{ on } \partial\Omega, \quad Eu = E^e + E^p, \quad \mathrm{curl} E^p \in L^2(\Omega, M^{3\times3}) \tag{31}$$

Remark 3. $E^e \in L^2(\Omega, M^{3\times3}_{sym})$ and $E^p \in BV(\Omega, M^{3\times3}_{sym}) \subset L^{3/2}(\Omega, M^{3\times3}_{sym})$ imply that $\Rightarrow\ Eu \in L^{3/2}(\Omega, M^{3\times3}_{sym})$. Thus, since $w \in H^1(\Omega,\mathbb{R}^3)$, it follows from Korn's inequality that $u \in W^{1,3/2}(\Omega,\mathbb{R}^3)$.
On the other hand, from the expression of the functional $\mathcal{H}$ and from the growth assumptions (25) on the derivatives of the function V, it is sufficient to find $\xi \in L^1(\Omega)$. However, such a choice might lead to solutions where ξ is a measure. To avoid this, we will consider here a perturbation of the model with the term $\beta|\nabla\xi|^2$ $(\beta > 0)$ for which $\xi \in H^1(\Omega)$.

The family of the admissible configurations for the boundary data w is denoted by

$$\mathbb{A}(w) := \{(u, E^e, E^p, \xi) \text{ such that (30)-(31) are satisfied}\} \quad (32)$$

The quasi-static evolution using global stability. The quasi-static evolution for the Gurtin-Anand's model using the global stability is defined as follows. Let $T > 0$. Find a map

$$t \to (u(t), E^e(t), E^p(t), \xi(t)) \in \mathbb{A}(w(t))$$

from $[0, T]$ to $W^{1,3/2}(\Omega, \mathbb{R}^3) \times L^2(\Omega, M^{3\times 3}_{sym}) \times BV(\Omega, M^{3\times 3}_{sym}) \times H^1(\Omega)$ such that

(a) *global stability:* for every $t \in [0, T]$ and for every $(v, E, P, \eta) \in \mathbb{A}(w(t))$

$$\begin{aligned} &\mathcal{Q}_1(E^e(t)) + \mathcal{Q}_2(E^p(t)) + \mathcal{V}(\xi(t)) + \frac{1}{2}\beta||\nabla\xi(t)||_0^2 \\ &\le \mathcal{Q}_1(E) + \mathcal{Q}_2(P) + \mathcal{V}(\eta) + \frac{1}{2}\beta||\nabla\eta||_0^2 - \mathcal{H}(P - E^p(t), \eta - \xi(t)) \end{aligned} \quad (33)$$

(b) *energy inequality:* for any $t \in [0, T]$

$$\mathcal{E}(t) + \mathcal{D}_{\mathcal{H}}(E^p, \xi; 0, t) \le \mathcal{E}(0) + \int_0^t \langle T(t) : E\dot{w}\rangle dt \quad (34)$$

where $\mathcal{E}(t) = \mathcal{Q}_1(E^e(t)) + \mathcal{Q}_2(\mathrm{curl} E^p(t)) + \mathcal{V}(\xi(t)) + \beta||\nabla\xi(t)||_0^2$
$T(t) = \mathcal{C}E^e(t)$ and $\mathcal{D}_{\mathcal{H}}$ is defined in (29).

Following [12, 13] the strategy to construct the evolution described above is to use a variational method based on time discretization and on the solutions of suitable incremental problems. Precisely, let $n \ge 1$ be an integer and let $t_n^i = \frac{i}{n}T$ with $i = 0, 1, \ldots, n$.
Set $u_{n,i} = u(t_n^i)$, $E^e_{n,i} = E^e(t_n^i)$, $E^p_{n,i} = E^p(t_n^i)$, $\quad \xi_{n,i} = \xi(t_n^i)$.
The incremental problem reads as follows: given $(u_{n,i}, E^e_{n,i}, E^p_{n,i}, \xi_{n,i}) \in \mathbb{A}(w(t_n^i))$, find $(u_{n,i+1}, E^e_{n,i+1}, E^p_{n,i+1}, \xi_{n,i+1}) \in \mathbb{A}(w(t_n^{i+1}))$ as the solution of the minimum problem

$$\min \left\{ \begin{array}{l} \mathcal{Q}_1(E) + \mathcal{Q}_2(\mathrm{curl} P) + \mathcal{V}(\eta) + \beta||\nabla\eta||_0^2 + \mathcal{H}(P - E^p_{n,i}, \eta - \xi_{n,i}), \\ (v, E, P, \eta) \in \mathbb{A}(w(t_n^{i+1}) \end{array} \right\} \quad (35)$$

However, since the energy functional in (35) is not convex, the global minimality criterion is not appropriate in this case. In fact, the lack of convexity usually allows the functional to have multiple wells. Therefore, a quasi-static evolution driven by a global minimality criterion might prescribe jumps on the solution from one well to another. So, we propose here a quasi-static evolution governed by a local minimality criterion. That is, we replace the global stability by a stationary point criterion which reads as follows: for any $t \in [0, T]$, there exist $T^p(t)$ second order tensor, $\mathbb{K}^p_{\mathrm{dis}}(t)$, and $\mathbb{K}^p(t)$ third order tensors such that, for $T(t) = \mathcal{C}E^e(t)$ and $g(t) = -V'(\xi(t)) + \beta\Delta\xi(t)$ we have

(a) $\mathrm{div}T(t) = 0$ in Ω and $\begin{cases} T^p(t) = T(t)_{\mathrm{D}} + \mathrm{div}\mathbb{K}^p(t) & \text{in } \Omega, \\ \mathbb{K}^p\nu = 0 \text{ on } \partial\Omega; \end{cases}$

(b) $\sqrt{|T^p(x,t)|^2 + \ell^{-2}|\mathbb{K}^p_{\mathrm{dis}}(x,t)|^2} \leq S_{\mathrm{Y}} - g(x,t)$ for a.e. $x \in \Omega$;

(c) $(\dot{E}^p(t,x), \nabla \dot{E}^p(t,x), \dot{\xi}(t,x)) \in N_{\mathcal{K}}(T^p(t,x), \mathbb{K}^p_{\mathrm{dis}}(t,x), g(t,x))$ where $\mathcal{K}$ is the set of generalized stresses defined in (13).

To construct the quasi-static evolution based of the stationary point criterion above and the energy inequality (34), we follow the vanishing viscosity approach in [3]. Precisely, we introduce an ε-regularized evolution.

The regularized evolution. Let $T > 0$ and $w \in H^1([0,T]; H^1(\Omega, \mathbb{R}^3)$. Let $(u_0, E^e_0, E^p_0, \xi_0) \in \mathbb{A}(w(0))$ and let $\varepsilon > 0$. A solution of the ε-regularized evolution in the time interval $[0,T]$ with boundary datum w and initial condition $(u_0, E^e_0, E^p_0, \xi_0)$ is a function $(u_\varepsilon, E^e_\varepsilon, E^p_\varepsilon, \xi_\varepsilon)$ with

$$\begin{aligned} &u_\varepsilon \in H^1([0,T]; W^{1,3/2}(\Omega, \mathbb{R}^3)), \quad && E^e_\varepsilon \in H^1([0,T]; L^2(\Omega, M^{3\times3}_{sym})) \\ &E^p_\varepsilon \in L^2([0,T]; BV(\Omega, M^{3\times3}_{sym})), \quad && \dot{E}^p_\varepsilon \in L^2(([0,T]; L^2(\Omega, M^{3\times3}_{sym})) \\ &\xi_\varepsilon \in H^1([0,T]; H^1(\Omega)) \end{aligned}$$

such that for every $t \in [0,T]$, there exists $T^p_\varepsilon(t)$ second order tensor, $\mathbb{K}^p_\varepsilon(t)$ and $\mathbb{K}^p_{\mathrm{dis},\varepsilon}(t)$ third order tensors such that, setting $g_\varepsilon(t) = -V'(\xi_\varepsilon(t)) + \beta\Delta\xi_\varepsilon(t)$ and $T_\varepsilon(t) = \mathcal{C}E^e_\varepsilon(t)$, we have

(a)$_\varepsilon$ $(u_\varepsilon(0), E^e_\varepsilon(0), E^p_\varepsilon(0), \xi_\varepsilon(0)) = (u_0, E^e_0, E^p_0, \xi_0)$;

(b)$_\varepsilon$ $\forall t \in [0,T]$, $(u_\varepsilon(t), E^e_\varepsilon(t), E^p_\varepsilon(t), \xi_\varepsilon(t)) \in \mathbb{A}(w(t))$;

(c)$_\varepsilon$ $\forall t \in [0,T]$, $\mathrm{div}T_\varepsilon(t) = 0$ in Ω and $\begin{cases} T^p_\varepsilon(t) = T(t)_{\varepsilon,\mathrm{D}} + \mathrm{div}\mathbb{K}^p_\varepsilon(t) & \text{in } \Omega, \\ \mathbb{K}^p_\varepsilon\nu = 0 \text{ on } \partial\Omega; \end{cases}$

(d)$_\varepsilon$ $\sqrt{|T^p_\varepsilon(x,t)|^2 + \ell^{-2}|\mathbb{K}^p_{\mathrm{dis},\varepsilon}(x,t)|^2} \leq S_{\mathrm{Y}} - g_\varepsilon(x,t)$ for a.e. $x \in \Omega$;

(e)$_\varepsilon$ $(\dot{E}^p_\varepsilon(t,x), \nabla\dot{E}^p_\varepsilon(t,x), \dot{\xi}_\varepsilon(t,x)) \in N^\varepsilon_{\mathcal{K}}(T^p_\varepsilon(t,x), \mathbb{K}^p_{\mathrm{dis},\varepsilon}(t,x), g_\varepsilon(t,x))$ where $N^\varepsilon_{\mathcal{K}}(\Sigma) = \frac{1}{\varepsilon}(\Sigma - P_{\mathcal{K}}(\Sigma))$, $P_{\mathcal{K}}$ is the projection onto $\mathcal{K}$.

The incremental problem. The incremental problem corresponding to the ε-regularized evolution above is given by

$$\min \begin{Bmatrix} \mathcal{Q}_1(E) + \mathcal{Q}_2(P) + \mathcal{V}(\eta) + \frac{1}{2}\beta||\nabla\eta||^2_0 + \mathcal{H}(P - E^p_{n,i}, \eta - \xi_{n,i}) \\ + \frac{n\varepsilon}{2T}||P - E^p_{n,i}||^2_0 + \frac{n\varepsilon}{2T}||\eta - \xi_{n,i}||^2_0, \qquad (v, E, P, \eta) \in \mathbb{A}(w(t^{i+1}_n)) \end{Bmatrix} \tag{36}$$

We prove in [5] that (36) has a unique solution for n large enough.

Now in the next step we define piecewise constant interpolants

$$u_{n,\varepsilon}(t) := u_{n,i},\ E^e_{n,\varepsilon}(t) := E^e_{n,i},\ E^p_{n,\varepsilon}(t) := E^p_{n,i},\ \xi_{n,\varepsilon}(t) := \xi_{n,i}$$

for every $t \in [t^i_n, t^{i+1}_n)$.

The ε-regularized evolution will then be obtained by taking the limit as $n \to \infty$ of these piecewise constant interpolants. The final step will be to study the asymptotic behaviour of the ε-regularized evolution as the viscosity parameter $\varepsilon \to 0$.

Remark 4. Following the paper [7], a further analysis would be on the asymptotic behaviour of the formulation above when the energetic and dissipative length scales L and ℓ go to zero.
On the other hand, it would be interesting to consider models involving the plastic spin.

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