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Some mathematical problems in perfect plasticity

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Abstract

Mathematical problems arising in the study of problems of perfect plasticity are reviewed, and some problems of current interest are discussed. In the context of the dual formulation, for which the problem has been thoroughly explored, the strategy of viscoplastic regularization and its application are reconsidered, with a view to exploring the use of this strategy in the study of the primal problem. It is argued that this approach is not without substantial difficulties, and so alternatives to the study of the primal problem are sought. Details are given of an approach by the authors in which direct methods of the calculus of variations are applied to a sequence of time-discrete problems, the solution to the original problem being recovered in the limit as the time step goes to zero.

Keyword: Perfect plasticity

1. Introduction

Qualitative analyses of the equations of elastoplasticity have enjoyed steady attention since the appearance of key works in this area in the 1970s. For the case of hardening plasticity a number of results have been obtained, and the theory for this class of problems is now well understood. The situation for perfect plasticity is different in that, while significant progress has been made, there remain some important open problems.

The variational problems for hardening and perfect plasticity differ in one major respect, in that the former is appropriately posed in a Hilbertian Sobolev space setting, while perfect plasticity demands that the displacement be sought in $BD(\Omega)$, the space of functions of bounded deformation. Physically, the difference is manifested in the ability of perfectly plastic materials to form shear bands, which are narrow bands of very high displacement gradients. The idealized situation is one in which a slip line occurs, and the displacement experiences a discontinuity in its tangential component.

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A great many situations exist in which the deformations of solid media are characterized by discontinuities either in the displacement, or in its derivatives with respect to space and time. Typical and common examples include fracture in solids, and delamination in laminated composites. A third example, particularly relevant to the present work, is that of localization in inelastic solids.

Key mathematical results for hardening plasticity are collected together in the monograph by Han and Reddy [11]. In this work, two alternative formulations of the variational problem are considered: the primal formulation, in which the unknown variables are the displacement and internal variables; and the dual formulation, in which the generalized stresses and velocity are the unknown variables. Results for the dual problem were earlier established by Johnson [18], while Han and Reddy [10] have carried out a complete analysis for the primal problem. The monograph [11] presents these and other results in a unified framework.

In the context of perfect plasticity there have been a number of investigations of well-posedness of the dual problem, and we mention in particular the early works of Duvaut and Lions [6] and Johnson [16], in which the stress was properly characterized but the displacement was not sought in the most appropriate setting. Matthies [22] and Suquet [33,34] presented analyses that established the existence of displacements in the space of functions of bounded deformation. Further analyses include those of Anzellotti [1], and Temam [36].

Anzellotti and Luckhaus [3] have studied the dynamic problem for perfect plasticity by approximating it through a sequence of problems for elastic-perfectly plastic materials with viscosity. The results for the original problems are recovered in the limit as the viscosity goes to zero.

Related studies of the Hencky problem, in which the rate formulation is replaced by a total strain problem, have been carried out by Anzellotti and Giaquinta [2] and Temam [35].

Analyses of finite element approximations of the initial-boundary value problem of elastoplasticity have enjoyed limited but steady attention. The first contribution was that of Johnson [17], who considered a formulation with stress as the primary variable, and who derived error estimates for the fully discrete problem. About the same time, independent work was carried out by Korneev and others [20,21], and by Hlaváček [14]; a summary account of the latter work may be found in [15]. Matthies [23] showed the convergence of semi-discrete finite element approximations of stress and displacement for the dual problem of perfect plasticity, but without explicit construction of the finite element spaces, nor with explicit rates of convergence. Johnson [19] subsequently analyzed fully discrete finite element approximations of the problem with hardening, in the context of a mixed formulation in which both stress and velocity are the variables.

The work by Han and Reddy [11] contains a detailed account of semi-and fully discrete approximations of the problem of hardening elastoplasticity, and error estimates for these approximations are derived there. Under assumptions of sufficient smoothness of the solution, optimal order error estimates are derived. In later work [12], the regularity assumptions are relaxed, and optimal error estimates are obtained under basic regularity conditions.

Computational studies of problems in perfect and softening plasticity offer a host of complexities, most of these arising from the need to accommodate sharp gradients and discontinuities in the solution. Solutions based on standard approaches exhibit a strong mesh dependence, and there have been many investigations aimed at circumventing this difficulty.

There have been extensive computational investigations of localization, manifested by the presence of shear bands, and Needleman and Tvergaard [25] have given a good overview of this line of study.

Simo et al. [32] address the issue of finite element approximations of problems involving rate-independent inelastic solids, in which strong discontinuities in the form of slip lines are present. A softening response is included in the model, and it is shown that the softening modulus is correctly interpreted in a distributional sense. One-dimensional finite element approximations based on regularizations of discontinuous interpolation functions verify the theoretical results.

The purpose of this work is, first, to provide an overview of some theoretical problems in perfect plasticity, with particular reference to considerations around the presence of discontinuities, and second, to highlight some current and recent developments. While there is an abundant computational literature on the problem, there remain many theoretical questions, some of which are central to a better understanding of computational approaches. For example, there exists no complete convergence analysis of finite element approximations for this class of problems.

In Section 2 the governing equations for elastic-perfectly plastic bodies are presented, and are cast in a form that exploits notions of convexity.

Before embarking on an analysis of the full set of governing equations, some aspects of a local analysis are presented, in order to highlight features associated with discontinuous solutions. This local analysis is presented in Section 3.

Section 4 is concerned with the dual formulation. Unlike the primal formulation there are a number of results available for the dual problem, and some strategies that have been successfully adopted in studying this problem are reviewed, partly to explore their applicability in studying the primal formulation. We pay particular attention to approaches based on viscoplastic regularization [34,36], and regularization through the introduction of hardening [5].

Next, we consider in Section 5 some approaches to studying the primal problem for perfect plasticity. The correct functional setting, that is, in which plastic strains and hence also total strains are bounded measures, rules out the possibility of formulating the problem directly as a variational inequality. We consider a variety of approaches, including that of viscoplastic regularization. In most cases it is not straightforward to carry over to the primal problem the strategies adopted successfully in respect of the dual problem, and we give an indication of the nature of these difficulties.

Section 6 is concerned with a new approach to the primal problem. This is based on the approach taken by Anzellotti and Giaquinta [2] for the Hencky problem. Here we show how the approach may be extended to the quasi-static problem by treating it as the limiting case of a sequence of Hencky-type problems, generated through a time discretization. The problem is then one of passing to the limit, as the time step goes to zero, and recovering in this limit the solution to the original problem.

2. The governing equations

Consider the initial-boundary value problem for quasi-static behavior of an elasto-plastic body which occupies a bounded domain $\Omega \subset \mathbb{R}^d$ ($d \leqslant 3$ for practical applications) with Lipschitz boundary Γ. We assume that deformations are sufficiently small to warrant adoption of the small strain assumption. The plastic behavior of the material is assumed to be describable within the classical framework of a convex, elastic domain coupled with the normality law. The yield surface, the boundary of the elastic domain, need not be smooth, however. The material is assumed to be perfectly plastic.

Suppose that the system is initially at rest, and that it is initially undeformed and unstressed. A time-dependent body force field $\boldsymbol{f}(\boldsymbol{x},t)$ is given, with $\boldsymbol{f}(\boldsymbol{x},0)=\mathbf{0}$. Then the problem is governed by the following set of equations in Ω:

the equilibrium equation

$$\operatorname{div} \boldsymbol{\sigma} + \boldsymbol{f} = \mathbf{0}; \tag{2.1}$$

the additive decomposition of strain

$$\boldsymbol{\epsilon} = \boldsymbol{e} + \boldsymbol{p}; \tag{2.2}$$

and the strain–displacement relation

$$\boldsymbol{\epsilon}(\boldsymbol{u}) = \tfrac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^{\mathrm{T}}). \tag{2.3}$$

Here $\boldsymbol{\sigma}$ is the stress tensor, $\boldsymbol{\epsilon}$ is the strain tensor, $\boldsymbol{u}$ the displacement vector, $\boldsymbol{p}$ the plastic strain tensor and $\boldsymbol{e}$ the elastic strain. All the tensors encountered here are symmetric. The plastic deformation is assumed to be incompressible so that

$$\mathrm{tr}\,\boldsymbol{p} = \boldsymbol{0} \quad \text{or} \quad p_{ii} = 0. \tag{2.4}$$

The summation convention for repeated indices is invoked here and henceforth.

For simplicity, and with little loss in generality, we take the boundary condition to be the homogeneous Dirichlet condition

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma, \tag{2.5}$$

while the initial conditions are assumed to be

$$\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{\sigma}(\boldsymbol{x}, 0) = \boldsymbol{0}. \tag{2.6}$$

A complete description of the problem requires that a set of constitutive equations be added to (2.1)–(2.6). The elastic relation is given by

$$\boldsymbol{\sigma} = \mathscr{C}\boldsymbol{e} = \mathscr{C}(\boldsymbol{\epsilon}(\boldsymbol{u}) - \boldsymbol{p}), \tag{2.7}$$

where $\mathscr{C}$ is the elasticity modulus. This tensor has the symmetry properties

$$C_{ijkl} = C_{jikl} = C_{klij}, \tag{2.8}$$

and the components are assumed to be bounded and measurable, that is,

$$C_{ijkl} \in L^{\infty}(\Omega). \tag{2.9}$$

In addition, the elasticity tensor is assumed to be pointwise stable; that is, there exists a constant $c_0 > 0$ such that

$$C_{ijkl}(x)\eta_{ij}\eta_{kl} \geqslant c_0|\boldsymbol{\eta}|^2 \text{ a.e. in } \Omega \tag{2.10}$$

for all symmetric $d \times d$ matrices $\boldsymbol{\eta}$.

The relation (2.7) has the inverse form

$$\boldsymbol{\epsilon}(\boldsymbol{u}) = \mathscr{A}\boldsymbol{\sigma} + \boldsymbol{p}, \tag{2.11}$$

in which $\mathscr{A}$, the inverse of $\mathscr{C}$, is the elastic compliance tensor.

For *isotropic* elastic materials the relation (2.7) becomes

$$\boldsymbol{\sigma} = \lambda(\mathrm{tr}\,\boldsymbol{e})\boldsymbol{I} + 2\mu\boldsymbol{e} := 2\mu\boldsymbol{e}^{\mathrm{D}} + d\kappa\boldsymbol{e}^{\mathrm{S}}. \tag{2.12}$$

Here $\boldsymbol{I}$ denotes the identity, $\boldsymbol{e}^{\mathrm{S}} := (1/d)(\mathrm{tr}\,\boldsymbol{e})\boldsymbol{I}$ is the spherical part of the tensor $\boldsymbol{e}$, and $\boldsymbol{e}^{\mathrm{D}} := \boldsymbol{e} - \boldsymbol{e}^{\mathrm{S}}$ is its deviatoric part. The material parameters λ and μ are the Lamé parameters, μ is also known as the shear modulus, and $\kappa := \lambda + (2/d)\mu$ is the bulk modulus.

Next, we set out the equations and inequalities governing plastic behaviour (see, for example, [11] for further details). The region of admissible stresses is the closed convex set given by

$$K = \{\boldsymbol{\sigma} \,|\, \phi(\boldsymbol{\sigma}) \leqslant 0, \quad \text{a.e. in } \Omega\}, \tag{2.13}$$

in which ϕ is a continuous convex function. The yield surface is the set of stresses satisfying $\phi(\boldsymbol{\sigma}(\boldsymbol{x}, t)) = 0$ for almost every $\boldsymbol{x}$ in Ω.

The flow law may then be expressed in the form

$$\dot{\boldsymbol{p}} \in N_K(\boldsymbol{\sigma}), \tag{2.14}$$

where $N_K(\boldsymbol{\sigma})$ denotes the normal cone to K at $\boldsymbol{\sigma}$. This expression may be written in the equivalent form

$$\dot{\boldsymbol{p}} \in \partial I_K(\boldsymbol{\sigma}), \tag{2.15}$$

where I_K is the indicator function of K, defined by

$$I_K(\boldsymbol{\sigma}) = \begin{cases} 0 & \text{if } \boldsymbol{\sigma} \in K, \\ +\infty & \text{otherwise.} \end{cases} \tag{2.16}$$

The subdifferential ∂F of any convex function $F : M^{d\times d} \to \mathbb{R} \cup \{\infty\}$ at $\boldsymbol{\sigma}$ is defined by

$$\partial F(\boldsymbol{\sigma}) = \{\boldsymbol{q} \,|\, F(\boldsymbol{\tau}) \geqslant F(\boldsymbol{\sigma}) + \boldsymbol{q} : (\boldsymbol{\tau} - \boldsymbol{\sigma}) \quad \text{for all } \boldsymbol{\tau} \in M^{d\times d}\}.$$

Equivalently, therefore, the flow law is given by

$$\dot{\boldsymbol{p}} : (\boldsymbol{\tau} - \boldsymbol{\sigma}) \leqslant 0 \quad \text{for all } \boldsymbol{\tau} \in K. \tag{2.17}$$

When the yield surface is smooth, the relation (2.14) together with (2.11) gives the set of relations

$$\begin{aligned} &\boldsymbol{\epsilon}(\dot{\boldsymbol{u}}) = \mathscr{A}\dot{\boldsymbol{\sigma}} + \lambda \boldsymbol{N}, \\ &\lambda \geqslant 0, \quad \phi(\boldsymbol{\sigma}) \leqslant 0, \quad \lambda\phi(\boldsymbol{\sigma}) = 0. \end{aligned} \tag{2.18}$$

The plastic strain rate is expressed here as a quantity that is parallel to the normal $\boldsymbol{N} := \partial\phi/\partial\boldsymbol{\sigma}$ to the yield surface. The plastic multiplier λ is non-negative, and strictly positive only if the yield condition is satisfied, when $\phi(\boldsymbol{\sigma}) = 0$.

An alternative way to describe the flow law is by using the support function of K, defined by

$$D(\dot{\boldsymbol{p}}) = \sup\{\dot{\boldsymbol{p}} : \boldsymbol{\tau} \,|\, \boldsymbol{\tau} \in K\}. \tag{2.19}$$

The function D is the Legendre–Fenchel conjugate of the indicator function; it is nonnegative and may take on the value $+\infty$. Then the flow law (2.14) is equivalent to the relation

$$\boldsymbol{\sigma} \in \partial D(\dot{\boldsymbol{p}}), \tag{2.20}$$

where the subdifferential $\partial D(\dot{\boldsymbol{p}})$ is the set of stresses $\boldsymbol{\sigma}$ satisfying

$$D(\boldsymbol{q}) \geqslant D(\dot{\boldsymbol{p}}) + \boldsymbol{\sigma} : (\boldsymbol{q} - \dot{\boldsymbol{p}}). \tag{2.21}$$

In the context of plasticity, the function D is a measure of the rate of irreversible or plastic work, and is known as the dissipation function.

We will refer later to the simple example of the von Mises yield criterion, for which the region of admissible stresses is given by

$$K = \{\boldsymbol{\tau} \,|\, \phi(\boldsymbol{\sigma}) := |\boldsymbol{\tau}^{\mathrm{D}}| - c_0 \leqslant 0\}, \tag{2.22}$$

where c_0 is a positive constant. For this yield criterion the inclusion (2.14) reads

$$\dot{\boldsymbol{p}} = \begin{cases} 0 & \text{if } \phi(\boldsymbol{\sigma}) < 0, \\ \lambda \dfrac{\boldsymbol{\sigma}^{\mathrm{D}}}{|\boldsymbol{\sigma}^{\mathrm{D}}|} & \text{otherwise,} \end{cases} \tag{2.23}$$

for some non-negative constant λ. The dissipation function is easily shown to be given by

$$D(\boldsymbol{q}) = \begin{cases} c_0|\boldsymbol{q}| & \text{if tr } \boldsymbol{q} = 0, \\ \infty & \text{otherwise.} \end{cases} \tag{2.24}$$

Two possible formulations of the initial-boundary value problem for elastoplasticity may be obtained from the set of equations summarized here: one, which we refer to as the *primal* formulation, is based on the flow law in the form (2.20), while the other is based on the dual form (2.14) of this law, and is therefore referred to as the *dual* formulation. Before considering these two formulations in greater detail we first take a closer look at local behavior for perfectly plastic bodies.

3. Local analysis

While the ultimate goal is that of understanding the global nature of discontinuous fields in plasticity, it is nevertheless instructive to carry out a local analysis, in which the consequences of a discontinuity at a point on a material surface $\mathscr{S}$ in Ω are explored. Such an analysis has been done by Simo et al. [32], whose motivation was the development of finite element approximations of discontinuous functions. We review some aspects of their illuminating analysis.

Discontinuous behavior across a surface in a body is usually characterized as *weak* if the displacement is continuous but the strain and velocity are discontinuous.

Let $\mathscr{S}$ be a smooth surface in Ω across which discontinuities occur. Denote by Ω^+ and Ω^- the two parts into which $\mathscr{S}$ divides Ω. Suppose that $\mathscr{S}$ is smooth, with unit normal $\boldsymbol{n}$ defined so that it points towards Ω^+.

Let ϕ be any function defined on the domain Ω, and assume that ϕ is continuous on $\Omega \setminus \mathscr{S}$. At any point $\boldsymbol{x}$ in $\mathscr{S}$, set $\phi^{\pm}(\boldsymbol{x}) = \lim_{s\to 0} \phi(\boldsymbol{x} \pm s\boldsymbol{n})$ (this limit is uniquely defined). The jump in ϕ at $\boldsymbol{x} \in \mathscr{S}$ is then defined by

$$[\![\phi]\!](\boldsymbol{x}) := \phi^+(\boldsymbol{x}) - \phi^-(\boldsymbol{x}). \tag{3.1}$$

Let $\boldsymbol{u}(\boldsymbol{x}, t)$ be the displacement field in Ω. A *weak discontinuity* on $\mathscr{S}$ is defined to be one for which

$$[\![\boldsymbol{u}]\!] = \boldsymbol{0} \quad \text{and} \quad [\![\nabla \boldsymbol{u}]\!] \neq \boldsymbol{0}, \tag{3.2}$$

at any point on $\mathscr{S}$. It follows that $[\dot{\boldsymbol{u}}] \neq \boldsymbol{0}$ also for a weak discontinuity.

On the other hand, a *strong discontinuity* is one for which

$$[\boldsymbol{u}] \neq \boldsymbol{0}. \tag{3.3}$$

Consider an elastic–perfectly plastic body on which the classical constitutive relations in Section 2 are applied. Assuming that the body force is continuous, equilibrium across the surface requires that

$$[\dot{\boldsymbol{\sigma}}]\boldsymbol{n} = \boldsymbol{0}.$$

Assume now that the displacement is of the form

$$\boldsymbol{u} = \boldsymbol{u}^{\mathrm{r}} + [\![\boldsymbol{u}]\!]H, \tag{3.4}$$

where $\boldsymbol{u}^{\mathrm{r}}$ is the regular or continuous part and H is the Heaviside step function with discontinuity located on the surface $\mathscr{S}$. The corresponding strain is then, formally,

$$\boldsymbol{\epsilon}(\boldsymbol{u}) = \boldsymbol{\epsilon}(\boldsymbol{u}^{\mathrm{r}}) + \mathrm{sym}([\![\boldsymbol{u}]\!] \otimes \boldsymbol{n})\delta, \tag{3.5}$$

in which δ is the Dirac delta. The use of $(2.18)_1$ in the equilibrium equation then leads to the condition

$$\boldsymbol{Q}^{\mathrm{e}}[\![\dot{\boldsymbol{u}}]\!] = \lambda^{\mathrm{s}}(\mathscr{C}\boldsymbol{N})\boldsymbol{n}, \tag{3.6}$$

in which λ^{s} is the singular part of the plastic multiplier, $\boldsymbol{N}$ is the normal to the yield surface, and $\boldsymbol{Q}^{\mathrm{e}}$ is the elastic acoustic tensor, given by

$$Q^{\mathrm{e}}_{ij}\boldsymbol{m} := \mathscr{C}_{ijkl}m_k m_l \quad \forall \boldsymbol{m} = \{m_k\}. \tag{3.7}$$

Finally, the use of the consistency condition $\dot{\phi} = 0$ provides a necessary condition, viz. that there exists $\boldsymbol{m} \neq 0$ such that

$$\boldsymbol{Q}^{\mathrm{ep}}\boldsymbol{m} = \boldsymbol{0}, \tag{3.8}$$

for the existence of a strong discontinuity, where $\boldsymbol{Q}^{\mathrm{ep}}$ is the elastic–plastic acoustic tensor defined by

$$\boldsymbol{Q}^{\mathrm{ep}} := \boldsymbol{Q}^{\mathrm{e}} - \frac{\mathscr{C}\boldsymbol{N} \otimes \mathscr{C}\boldsymbol{N}}{\boldsymbol{N} : \mathscr{C}\boldsymbol{N}}. \tag{3.9}$$

The condition for localization is therefore that there exist directions $\boldsymbol{m}$ such that

$$\det\ \boldsymbol{Q}^{\mathrm{ep}}\boldsymbol{m} = 0. \tag{3.10}$$

Conditions similar to (3.10) have earlier been obtained as conditions for the existence of *weak* discontinuities (see, for example, the seminal work of Hill [13], and the overview [25] by Needleman and Tvergaard). They have also served as the point of departure for computational studies of strong discontinuities by Simo et al. [32], who developed the theory in one space dimension with an extension to higher dimensions, and by Oliver [26,27] and Armero and Garikipati [4] who developed further the two-dimensional case.

4. The dual formulation

We return now to the quasi-static problem formulated on the domain $\overline{\Omega} \times [0, T]$, and focus on the dual problem. The point of departure is the flow law in the form (2.14) or (2.15). The plastic strain rate is eliminated as a variable by making use of the constitutive law (2.11) in rate form together with (2.15); this gives

$$(\boldsymbol{\epsilon}(\dot{\boldsymbol{u}}) - \mathscr{A}\dot{\boldsymbol{\sigma}}) : (\boldsymbol{\tau} - \boldsymbol{\sigma}) \leqslant 0 \quad \text{for all } \boldsymbol{\tau} \in K. \tag{4.1}$$

The dual problem then consists of finding $\boldsymbol{u}$ and $\boldsymbol{\sigma}$ that satisfy (2.1), (2.5), (2.6), with (2.7).

This problem may be cast in weak or variational form by first introducing spaces V of displacements and S of stresses. We postpone for the moment the definition of V, assuming only that it is such that the weak problem makes sense, and we define the spaces S and $\mathscr{P}$ of stresses and admissible stresses by

$$\begin{aligned} &S = \{\boldsymbol{\tau} = (\tau_{ij}) \mid \tau_{ji} = \tau_{ij}, \quad \tau_{ij} \in L^2(\Omega)\}, \\ &\mathscr{P} = \{\boldsymbol{\tau} \in S \mid \boldsymbol{\tau} \in K \quad \text{a.e. in } \Omega\}. \end{aligned} \tag{4.2}$$

Further background on function spaces that are relevant to the problem of perfect plasticity may be found in the Appendix A.

Denoting by $(\cdot,\cdot)$ the L^2 inner product on Ω, the weak problem is one of finding $\boldsymbol{u} : [0, T] \to V$ and $\boldsymbol{\sigma} : [0, T] \to \mathscr{P}$ with $\boldsymbol{u}(0) = \boldsymbol{0}$ and $\boldsymbol{\sigma}(0) = \boldsymbol{0}$ that satisfy

$$\begin{aligned} &(\boldsymbol{\sigma}, \boldsymbol{\epsilon}(\boldsymbol{v})) = \langle \ell(t), \boldsymbol{v} \rangle, \\ &-(\mathscr{A}\dot{\boldsymbol{\sigma}}, \boldsymbol{\tau} - \boldsymbol{\sigma}) + (\boldsymbol{\epsilon}(\dot{\boldsymbol{u}}), \boldsymbol{\tau} - \boldsymbol{\sigma}) \leqslant 0 \end{aligned} \tag{4.3}$$

for all $\boldsymbol{v}, \boldsymbol{\tau} \in V \times \mathscr{P}$. The formal equivalence of problem (4.3) to the set of governing equations is clear.

The dual problem may be reduced to one involving the stress only, by introducing the time-dependent constraint set

$$\mathscr{P}(t) = \{\boldsymbol{\tau} \in \mathscr{P} \,|\, (\boldsymbol{\tau}, \boldsymbol{\epsilon}(\boldsymbol{u})) = \langle \ell(t), \boldsymbol{v} \rangle \quad \text{for all } \boldsymbol{v} \in V\}. \tag{4.4}$$

Then the reduced problem is one of seeking $\boldsymbol{\sigma} : [0, T] \to \mathscr{P}(t)$ that satisfies

$$(\mathscr{A}\dot{\boldsymbol{\sigma}}, \boldsymbol{\tau} - \boldsymbol{\sigma}) \geqslant 0 \quad \text{for all } \boldsymbol{\tau} \in \mathscr{P}(t). \tag{4.5}$$

The problems (4.3) and (4.5) have been given a detailed treatment by Johnson [16], Matthies [22], and Suquet [33,34], for the case of perfect plasticity, and by Johnson [18] for the case of hardening.

For the case of hardening plasticity, the space V is simply the Sobolev space $[H_0^1(\Omega)]^d$. But for the case of perfect plasticity, for which the function space setting must make provision for the presence of spatial discontinuities in displacement, a Sobolev space setting is inadequate. Instead, we require the space $\mathrm{BD}(\Omega)$ of functions of bounded deformation, which is the space of integrable displacements whose strains are bounded measures (see Appendix A). This setting is able to accommodate displacement fields that include slip lines, for example.

For the perfectly plastic problem we define the space V of displacements by

$$V = \mathrm{BD}(\Omega). \tag{4.6}$$

4.1. Viscoplastic regularization

The problem of existence of solutions to the dual problem can be approached by regularizing this problem in such a way as to obtain a problem with solutions in 'nice' spaces, then to show that the regularized solution converges in some sense to functions that solve the original problem. This approach has been employed by Johnson [16], Suquet [34], and Temam [36], in studies of perfectly plastic problems.

The essence of the process is to replace the indicator function in (2.14) by a function γ that is differentiable, and that converges in some sense to the indicator function. The elastic relation then becomes

$$\mathscr{A}\dot{\boldsymbol{\sigma}}_\mu + \gamma'_\mu(\boldsymbol{\sigma}_\mu) = \boldsymbol{\epsilon}(\dot{\boldsymbol{u}}_\mu), \tag{4.7}$$

and one recovers (2.14) in the limit, as $\mu \to 0$.

The regularized problem is one of finding $\boldsymbol{\sigma}_\mu(t) \in S$ and $\boldsymbol{u}_\mu(t) \in V_\mu$ that satisfy the equilibrium equation, the boundary and initial conditions, and (4.7), and where V_μ is chosen appropriately.

The strategy then takes the following form:

(a) establish existence and uniqueness of a solution $(\boldsymbol{\sigma}_\mu, \boldsymbol{u}_\mu)$ to the regularized problem, for any μ;
(b) show that the solutions $\boldsymbol{\sigma}_\mu$ and $\boldsymbol{u}_\mu$ are bounded, independent of μ, in appropriate spaces;
(c) use the above boundedness and the compactness properties of the spaces concerned to deduce the existence of subsequences that converge to limits $(\boldsymbol{\sigma}, \boldsymbol{u})$;
(d) verify that the limits of the subsequences in fact solve the original problem.

The precise form of the space V_α will depend on the choice of regularization. We now take a closer look at two popular choices, viz. those attributed to Perzyna and to Norton and Hoff.

4.2. The Perzyna regularization [28]

Given $\mu > 0$, the potential for viscoplastic regularization in the case of the Perzyna law is

$$\gamma_\mu(\boldsymbol{\tau}) := \frac{1}{2\mu}|\boldsymbol{\tau} - \Pi_K(\boldsymbol{\tau})|^2, \tag{4.8}$$

where K, as before, denotes the region of admissible stresses and Π_K is the orthogonal projection onto K. This potential is indeed differentiable, with (Fréchet) derivative given by

$$\gamma'_\mu(\boldsymbol{\tau}) = \frac{1}{\mu}(\boldsymbol{\tau} - \Pi_K(\boldsymbol{\tau})). \tag{4.9}$$

For the von Mises yield condition one has

$$\Pi_K(\boldsymbol{\tau}) = \begin{cases} \boldsymbol{\tau}^{\mathrm{S}} + c_0 \dfrac{\boldsymbol{\tau}^{\mathrm{D}}}{|\boldsymbol{\tau}^{\mathrm{D}}|} & \text{if } |\boldsymbol{\tau}^{\mathrm{D}}| > c_0, \\ \boldsymbol{\tau} & \text{otherwise.} \end{cases} \tag{4.10}$$

4.3. The Norton–Hoff regularization

This form, introduced by Friaâ [9], is constructed by using the gauge function g of K, defined by [11]

$$g(\boldsymbol{\tau}) = \inf\{\alpha \geqslant 0 \,|\, \boldsymbol{\tau} \in \alpha K\}. \tag{4.11}$$

The gauge of K is convex and positively homogeneous of degree one. The Norton–Hoff regularization is then

$$g_p(\boldsymbol{\tau}) = \frac{1}{p'}[g(\boldsymbol{\tau})]^{p'}, \tag{4.12}$$

where $p' = p/(p-1)$. The perfectly plastic law is recovered in the limit as $p' \to \infty$.

It is of interest also to consider a modification of this regularization due to Temam [36]. Let $d(\boldsymbol{\tau})$ denote the distance from $\boldsymbol{\tau} \in E$ to K, and set

$$\theta(\boldsymbol{\tau}) = (1 + d^2(\boldsymbol{\tau}))^{1/2}.$$

Note that $\theta(\boldsymbol{\tau}^{\mathrm{D}}) = \theta(\boldsymbol{\tau})$ and also that $\theta(\boldsymbol{\tau}) = 1$ if and only if $\boldsymbol{\tau} \in K$. Define γ_μ by

$$\gamma_\mu(\boldsymbol{\tau}) = \frac{\mu}{\mu+1}\theta(\boldsymbol{\tau})^{(1+\mu)/\mu}. \tag{4.13}$$

Then γ_μ is differentiable, with derivative given by

$$\gamma'_\mu(\boldsymbol{\tau}) = (1 + d^2(\boldsymbol{\tau}))^{(1-\mu)/2\mu}(\boldsymbol{\tau} - \Pi_K\boldsymbol{\tau}). \tag{4.14}$$

Temam [36] has carried out a detailed analysis of the quasi-static problem for perfect plasticity, using the Norton–Hoff regularization. This is done by first showing that the regularized problem has a unique solution $\boldsymbol{\sigma}_\mu(t)$ with $\boldsymbol{\sigma}^{\mathrm{D}}_\mu(t) \in L^{(1+\mu)/\mu}(\Omega)$ and $\boldsymbol{u}_\mu(t) \in W^{1,1+\mu}(\Omega)$. Then the solution is shown to satisfy various bounds independent of μ, and the weak compactness of the relevant spaces leads to the existence of weak limits $\boldsymbol{\sigma}$ and $\boldsymbol{u}$. The stress $\boldsymbol{\sigma}(t)$ is shown to belong to L^2, and is continuous in time, while the displacement $\boldsymbol{u}$ belongs to $\mathrm{BD}(\Omega)$, with $\operatorname{div}\boldsymbol{u} \in L^2(\Omega)$.

A key contribution of Temam [36] was to give a rigorous sense to the flow law, in particular the term $\epsilon(\dot{\boldsymbol{u}}) : \boldsymbol{\tau}$, which is shown to be a bounded measure. The flow law is then shown to be satisfied in the sense of measures.

Other important contributions that make use of regularization are those of Suquet [33,34] and Johnson [16].

4.4. Safe load condition

This is a key condition in all existence proofs for the dual problem. It can be formulated in various alternative ways, for example, as the requirement that there exist a stress $\boldsymbol{\sigma}^*$ with the property that $\boldsymbol{\sigma}^*$ and $\boldsymbol{\sigma}^*_t$ are bounded functions, that $\boldsymbol{\sigma}^*$ in addition satisfies the equilibrium equation, and that

$$\boldsymbol{\sigma}^*(\boldsymbol{x}, t) \in K, \quad \operatorname{dist}(\boldsymbol{\sigma}^*(\boldsymbol{x}, t), \partial K) \geqslant c > 0 \ \text{ for all } \boldsymbol{x}, t, \tag{4.15}$$

where as before K is the region of admissible stresses and ∂K is the yield surface. The function $\operatorname{dist}(\cdot,\cdot)$ is the normed distance between a point and a set.

5. The primal problem

While significant advances have been made in investigating the dual problem, there has not been the same degree of progress with the primal problem. Part of the reason is that the primal formulation is more recent, having first been presented in [29]. A second consideration is that the dual form is more prevalent in applications, and has in fact been the basis of most of the work aimed at developing and implementing efficient solution algorithms (see, for example, [31]). Nevertheless the primal problem is of interest in its own right; it is a kinematically based formulation, with unknowns being the displacement, plastic strain and, in the case of hardening, the internal variables. Moreover, it reduces in a natural way to the displacement problem for elasticity.

We begin by considering the appropriate setting for this problem. Let

$$Q = \{\boldsymbol{q} = (q_{ij})_{d\times d} : q_{ji} = q_{ij},\ q_{ij} \in \mathscr{M}(\Omega)\}, \tag{5.1}$$

where $\mathscr{M}(\Omega)$ denotes the space of bounded Radon measures on Ω, defined in the Appendix A. The space Q_0 of plastic strains is then defined to be the closed subspace of Q defined by

$$Q_0 = \{\boldsymbol{q} \in Q : \operatorname{tr} \boldsymbol{q} = 0\}. \tag{5.2}$$

As before, we require that $V = \mathrm{BD}(\Omega)$. Bearing in mind that $\boldsymbol{\epsilon}(\boldsymbol{u}) - \boldsymbol{p}$ represents the elastic strains, which are expected to be regular, we may define the product space Z of displacements and plastic strains by

$$Z = \{(\boldsymbol{v}, \boldsymbol{q}) | \boldsymbol{v} \in \mathrm{BD}(\Omega),\ \boldsymbol{q} \in Q_0,\ \boldsymbol{\epsilon}(\boldsymbol{v}) - \boldsymbol{q} \in \boldsymbol{L}^2(\Omega)\}. \tag{5.3}$$

The natural next step would be to formulate the problem in the form of a variational inequality, as has been done for the case of hardening. This approach is problematic, and we will show why this is so, by looking at the variational formulation. For the moment, we proceed purely formally, that is, without regard to the spaces of displacements and plastic strains.

We introduce the bilinear form

$$a(\boldsymbol{w}, \boldsymbol{z}) = \int_\Omega \mathscr{C}(\boldsymbol{\epsilon}(\boldsymbol{u}) - \boldsymbol{p}) : (\boldsymbol{\epsilon}(\boldsymbol{v}) - \boldsymbol{q})\, \mathrm{d}\boldsymbol{x}, \tag{5.4}$$

a linear functional

$$\langle \ell(t), \boldsymbol{z}\rangle = \int_\Omega \boldsymbol{f}(t) \cdot \boldsymbol{v}\, \mathrm{d}\boldsymbol{x}, \tag{5.5}$$

and a functional

$$j(\boldsymbol{z}) = \int_\Omega D(\boldsymbol{q})\, \mathrm{d}\boldsymbol{x}, \tag{5.6}$$

where $\boldsymbol{w} = (\boldsymbol{u}, \boldsymbol{p})$ and $\boldsymbol{z} = (\boldsymbol{v}, \boldsymbol{q})$.

From the properties of D, we see that $j(\cdot)$ is a convex, positively homogeneous, nonnegative and l.s.c. functional.

Then the primal variational problem of elastoplasticity is as follows: find $\boldsymbol{w}(t) = (\boldsymbol{u}(t), \boldsymbol{p}(t))$ such that for almost all $t \in (0, T)$,

$$a(\boldsymbol{w}(t), \boldsymbol{z} - \dot{\boldsymbol{w}}(t)) + j(\boldsymbol{z}) - j(\dot{\boldsymbol{w}}(t)) \geqslant \langle \ell(t), \boldsymbol{z} - \dot{\boldsymbol{w}}(t)\rangle. \tag{5.7}$$

There is a fundamental difficulty in considering the primal problem in the form (5.7). This has to do with the fact that the integrand of $a(\cdot,\cdot)$ in the variational inequality contains terms of the form $\boldsymbol{\epsilon}(\boldsymbol{u}) : \boldsymbol{\epsilon}(\boldsymbol{v})$ and

$\epsilon(\boldsymbol{u}) : \boldsymbol{p}$. In both cases we are dealing with terms that involve the product of measures, and the meaning of such terms is not clear.

One therefore has to consider alternative approaches to the primal problem. A first alternative involves regularization of the kind that was discussed in Section 4. This can be achieved in two possible ways:

(a) by regularizing the dual problem, as has been done earlier, and then dualizing this to get a regular primal problem; or
(b) by carrying out a direct regularization of the non-differentiable term $j(\cdot)$ in the primal problem.

These two approaches will lead to two different problems. Either way, though, they rely on an ability to show existence and uniqueness in suitable Lebesgue or Sobolev spaces of solutions to the regularized problem, then to establish a priori bounds that will lead to the existence of weak limits. The corresponding total and plastic strains would be measures, and the remaining task would be one of showing that these limits satisfy the primal problem in a suitable sense. We now take a look at these two approaches.

5.1. Dualizing the regularized problem

We return to (2.14) and (2.20), which are duals of each other. The regularized form of (2.14) is

$$\dot{\boldsymbol{p}}_\mu = \gamma'_\mu(\boldsymbol{\sigma}_\mu), \tag{5.8}$$

and the dual of this inclusion is

$$\boldsymbol{\sigma}_\mu = (\gamma^*_\mu)'(\dot{\boldsymbol{p}}_\mu), \tag{5.9}$$

where f^* denotes the Legendre–Fenchel dual of a convex function f [8]. Thus we have to determine γ^*_μ. We do this for the two regularizations introduced earlier, in the context of the von Mises criterion.

First, for the Perzyna relation the Legendre–Fenchel dual of γ_μ is given by

$$\gamma^*_\mu(\boldsymbol{q}) := \sup_{\boldsymbol{\tau}} \left[\boldsymbol{\tau} : \boldsymbol{q} - \frac{1}{2\mu}|\boldsymbol{\tau} - \Pi_K(\boldsymbol{\tau})|^2\right]. \tag{5.10}$$

Note that

$$\gamma^*_\mu(\boldsymbol{q}) = \max\left(I_1 := \sup_{\boldsymbol{\tau}\in K}[\boldsymbol{\tau} : \boldsymbol{q}], I_2 := \sup_{\boldsymbol{\tau}\notin K}\left[\boldsymbol{\tau} : \boldsymbol{q} - \frac{1}{2\mu}|\boldsymbol{\tau} - \Pi_K(\boldsymbol{\tau})|^2\right]\right). \tag{5.11}$$

It is easy to see that I_1 in (5.11) is equal to $c_0|\boldsymbol{q}^{\mathrm{D}}|$. Let us now compute I_2.

From the decomposition into spherical and deviatoric parts, we have

$$\begin{aligned} I_2 &= \sup_{\alpha\in\mathbb{R}} \alpha\,\mathrm{tr}(\boldsymbol{q}) + \sup_{\xi\in Q_0, |\xi|>c_0}\left[\boldsymbol{\xi} : \boldsymbol{q}^{\mathrm{D}} - \frac{1}{2\mu}\left|\boldsymbol{\xi} - c_0\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right|^2\right] \\ &= \begin{cases} \sup\limits_{\xi\in Q_0, |\xi|>c_0}\left[\boldsymbol{\xi} : \boldsymbol{q} - \frac{1}{2\mu}\left|\boldsymbol{\xi} - c_0\frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|}\right|^2\right] & \text{if } \boldsymbol{q}\in Q_0, \\ +\infty & \text{otherwise.} \end{cases} \end{aligned}$$

So, if $\boldsymbol{q} \in Q_0$, we get

$$\begin{aligned} I_2 &= \sup_{r>c_0} \sup_{\xi \in Q_0, |\xi|=r} \left[\boldsymbol{\xi} : \boldsymbol{q} - \frac{1}{2\mu} \left| \boldsymbol{\xi} - c_0 \frac{\boldsymbol{\xi}}{|\boldsymbol{\xi}|} \right|^2 \right] \\ &= \sup_{r>c_0} \left[r|\boldsymbol{q}| - \frac{1}{2\mu}(r^2 - 2rc_0 + c_0^2) \right] \\ &= \frac{\mu}{2}|\boldsymbol{q}|^2 + c_0|\boldsymbol{q}|. \end{aligned}$$

Coming back to (5.11), we obtain, for $\boldsymbol{q} \in Q_0$,

$$\gamma_\mu^*(\boldsymbol{q}) = \frac{\mu}{2}|\boldsymbol{q}|^2 + c_0|\boldsymbol{q}|. \tag{5.12}$$

It is worth noting that the dual potential γ_μ^*, which can be interpreted as a regularized dissipation function, differs from the dissipation function (2.24) by a term $\frac{1}{2}\mu|\boldsymbol{q}|^2$ that corresponds to the presence of linear kinematic hardening. Thus, this form of regularization is equivalent to regularization by hardening.

In the same way it can be shown, for example, that the dual of the regularized Norton–Hoff potential (4.12) is given by

$$g_p^*(\boldsymbol{q}) = \frac{1}{p}(c_0|\boldsymbol{q}|)^p. \tag{5.13}$$

We recall that, in order to recover the case of perfect plasticity, we let $p \to 1$, or $p' \to \infty$.

The regularized primal problem. We now explore the feasibility of treating the primal problem by dualization of the regularized dual problem, and show that such an approach presents non-trivial difficulties. For the Perzyna approach, set

$$j_\mu(\boldsymbol{z}) := \int_\Omega \gamma_\mu^*(\boldsymbol{q})\, \mathrm{d}\boldsymbol{x} \quad \text{for } \boldsymbol{z} = (\boldsymbol{v}, \boldsymbol{q}),$$

where γ_μ^* is obtained from the von Mises condition, and is defined in (5.12), and consider the primal problem of finding $\boldsymbol{w}_\mu(t) = (\boldsymbol{u}_\mu(t), \boldsymbol{p}_\mu(t))$ such that for almost all $t \in (0, T)$,

$$a(\boldsymbol{w}_\mu(t), \boldsymbol{z} - \dot{\boldsymbol{w}}_\mu(t)) + j_\mu(\boldsymbol{z}) - j_\mu(\dot{\boldsymbol{w}}_\mu(t)) \geqslant \langle \ell(t), \boldsymbol{z} - \dot{\boldsymbol{w}}_\mu(t) \rangle. \tag{5.14}$$

That is, we seek

$$\boldsymbol{u}_\mu \in H^1((0,T); \boldsymbol{H}^1(\Omega)) \quad \text{and} \quad \boldsymbol{p}_\mu \in H^1((0,T); \boldsymbol{L}^2(\Omega)),$$

such that for almost all $t \in (0, T)$, $\dot{\boldsymbol{p}}_\mu(t) \in Q_0$, $\boldsymbol{u}_\mu(t) = 0$ on Γ, and

$$\begin{aligned} &\int_\Omega \mathscr{C}(\boldsymbol{\epsilon}(\boldsymbol{u}_\mu) - \boldsymbol{p}_\mu) : ((\boldsymbol{\epsilon}(\boldsymbol{v}) - \boldsymbol{q}) - (\boldsymbol{\epsilon}(\dot{\boldsymbol{u}}_\mu) - \dot{\boldsymbol{p}}_\mu))\, \mathrm{d}\boldsymbol{x} \\ &\quad + \frac{\mu}{2}\int_\Omega |\boldsymbol{q}|^2\, \mathrm{d}\boldsymbol{x} + c_0 \int_\Omega |\boldsymbol{q}|\, \mathrm{d}\boldsymbol{x} - \frac{\mu}{2}\int_\Omega |\dot{\boldsymbol{p}}_\mu|^2\, \mathrm{d}\boldsymbol{x} - c_0 \int_\Omega |\dot{\boldsymbol{p}}_\mu|\, \mathrm{d}\boldsymbol{x} \\ &\quad \geqslant \int_\Omega \boldsymbol{f}(t)(\boldsymbol{v} - \dot{\boldsymbol{u}}_\mu(t))\, \mathrm{d}\boldsymbol{x} \quad \forall (\boldsymbol{v}, \boldsymbol{p}) \in \boldsymbol{H}^1(\Omega) \times \boldsymbol{L}^2(\Omega). \end{aligned} \tag{5.15}$$

Here there are two main handicaps:

- the regularized dissipation functional j_μ is not homogeneous of degree one;
- the bilinear form a is only coercive in the difference $\boldsymbol{\epsilon}(\boldsymbol{v}) - \boldsymbol{p}$, and not in $\boldsymbol{v}$ and $\boldsymbol{p}$ separately.

So it is not possible to make progress using the strategy in [11] based on time discretization. Therefore, even the existence of the regularized solution $(\boldsymbol{u}_\mu, \boldsymbol{p}_\mu)$ is not guaranteed in this setting.

5.2. Direct regularization

In this approach one would regularize the dissipation function. There there are various ways of doing this, for example,

$$D^1_\mu(\boldsymbol{q}) := (D * \rho_\mu)(\boldsymbol{q}), \quad D^2_\mu(\boldsymbol{q}) := c_0\sqrt{\mu^2 + |\boldsymbol{q}|^2}, \quad D^3_\mu(q) := \frac{c_0|\boldsymbol{q}|^2}{\sqrt{\mu^2 + |\boldsymbol{q}|^2}}, \tag{5.16}$$

where $*$ denotes convolution and (ρ_μ) is a mollifier or smoothing function. All these regularized dissipations are convex, non-homogeneous, and converge pointwise to D as $\mu \to 0$. Hence even by direct regularization of the dissipation potential, one cannot obtain the existence of the regularized solution following the approach in [11], in which the one-homogeneity of the dissipation is a fundamental ingredient.

5.3. An approach to solving the dynamic problem

When regularizing perfect plasticity problems, even if the solutions of the regularized problems are obtained in L^p ($p > 1$) or Sobolev spaces, one would need some uniform (with respect to the regularization parameter) L^1-estimates for the total stain or the plastic stain. Then the compactness would provide the solution of the original problem, as a weak $*$ limit of the regularized solutions (see Appendix A).

We look at one such approach, due to Chełmiński [5], in the context of the *dynamic* problem. Here hardening regularization together with a further regularization to achieve coerciveness are used successfully to solve the dynamic problem for perfect plasticity.

The following problem is considered: find the velocity field $\boldsymbol{v} : \Omega \times (0, T) \to \mathbb{R}^3$, the infinitesimal strain $\boldsymbol{\epsilon} : \Omega \times (0, T) \to M^{3\times3}_s$, and the plastic strain $\boldsymbol{p} : \Omega \times (0, T) \to M^{3\times3}_s$ which satisfy the system of equations

$$\begin{aligned}
&\rho \boldsymbol{v}_t = \operatorname{div} \mathscr{C}(\boldsymbol{\epsilon} - \boldsymbol{p}) + \boldsymbol{f} \quad \text{in } \Omega \times (0, T),\\
&\boldsymbol{\epsilon}_t = \tfrac{1}{2}(\nabla \boldsymbol{v} + \nabla^{\mathrm{T}} \boldsymbol{v}),\\
&\boldsymbol{p}_t \in \partial I_K(\boldsymbol{\sigma}),\\
&\boldsymbol{v} = \boldsymbol{g} \quad \text{on } \partial\Omega \times (0, T),\\
&\boldsymbol{v}(\boldsymbol{x}, 0) = \boldsymbol{v}^0(\boldsymbol{x}), \quad \boldsymbol{\epsilon}(\boldsymbol{x}, 0) = \boldsymbol{\epsilon}^0(\boldsymbol{x}) \quad \boldsymbol{p}(\boldsymbol{x}, 0) = \boldsymbol{p}^0(\boldsymbol{x}).
\end{aligned} \tag{5.17}$$

Here a subscript t denotes differentiation with respect to time, $\boldsymbol{g} : \partial\Omega \times (0, T) \to \mathbb{R}^3$ is the boundary data and $\boldsymbol{v}^0 : \Omega \to \mathbb{R}^3$, $\boldsymbol{p}^0 : \Omega \to M^{3\times3}_s$ are the initial data.

The associated free energy function is

$$\rho\psi(\boldsymbol{\epsilon}, \boldsymbol{p}) := \tfrac{1}{2}\mathscr{C}(\boldsymbol{\epsilon} - \boldsymbol{p}) : (\boldsymbol{\epsilon} - \boldsymbol{p}),$$

which is only positive semi-definite; in fact it is zero when $\boldsymbol{p} = \boldsymbol{\epsilon}$. Hence the model is monotone but not coercive. The other difficulty in studying in this problem concerns the non-smoothness of the inelastic potential I_K.

The strategy in [5] is to approximate (5.17) by a model of perfect plasticity with isotropic hardening, viz.

$$\begin{aligned}
&\rho \boldsymbol{v}_t = \operatorname{div} \mathscr{C}(\boldsymbol{\epsilon} - \boldsymbol{p}) + \boldsymbol{f} \quad \text{in } \Omega \times (0, T),\\
&\boldsymbol{\epsilon}_t = \tfrac{1}{2}(\nabla \boldsymbol{v} + \nabla^{\mathrm{T}} \boldsymbol{v}),\\
&\boldsymbol{p}_t \in \partial I_{K(y)}(\boldsymbol{\sigma}),\\
&y_t = \eta|\boldsymbol{p}_t|,\\
&\boldsymbol{v} = \boldsymbol{g} \quad \text{on } \partial\Omega \times (0, T),\\
&\boldsymbol{v}(\boldsymbol{x}, 0) = \boldsymbol{v}^0(\boldsymbol{x}), \quad \boldsymbol{\epsilon}(\boldsymbol{x}, 0) = \boldsymbol{\epsilon}^0(\boldsymbol{x}), \quad \boldsymbol{p}(\boldsymbol{x}, 0) = \boldsymbol{p}^0(\boldsymbol{x}), \quad y(\boldsymbol{x}, 0) = y^0(\boldsymbol{x}),
\end{aligned} \tag{5.18}$$

where y is a scalar internal variable describing the isotropic hardening of the material, $\eta > 0$ is a material constant, and

$$K(y) := \{\boldsymbol{\sigma} \in M_s^{3\times 3} : |\boldsymbol{\sigma}^{\mathrm{D}}| \leqslant c_0 + \eta y\}. \tag{5.19}$$

However, even for this model the free energy

$$\rho\psi(\boldsymbol{\epsilon}, \boldsymbol{p}, y) := \tfrac{1}{2}\mathscr{C}(\boldsymbol{\epsilon} - \boldsymbol{p}) : (\boldsymbol{\epsilon} - \boldsymbol{p}) + \tfrac{1}{2}|y|^2 \tag{5.20}$$

is only positive semi-definite and hence the model is non-coercive; so the task of showing global existence of a solution in time is not straightforward. Thus the problem is regularized again, by a sequence of problems in which a term is added to achieve coerciveness. This results in the problem

$$\begin{aligned}
&\rho \boldsymbol{v}_t^k = \operatorname{div} \mathscr{C}\left(\boldsymbol{\epsilon}^k - \boldsymbol{p}^k + \frac{1}{k}\boldsymbol{\epsilon}^k\right) + \boldsymbol{f} \quad \text{ in } \Omega \times (0, T),\\
&\boldsymbol{\epsilon}_t^k = \tfrac{1}{2}(\nabla \boldsymbol{v}^k + \nabla^{\mathrm{T}} \boldsymbol{v}^k),\\
&(\boldsymbol{p}_t^k, y_t^k) \in \partial I_{\mathscr{K}}(\mathscr{C}(\boldsymbol{\epsilon}^k - \boldsymbol{p}^k), -y^k),\\
&\boldsymbol{v}^k = \boldsymbol{g} \quad \text{on } \partial\Omega \times (0, T),\\
&\boldsymbol{v}^k(\boldsymbol{x}, 0) = \boldsymbol{v}_k^0(\boldsymbol{x}), \quad \boldsymbol{\epsilon}^k(\boldsymbol{x}, 0) = \boldsymbol{\epsilon}_k^0(\boldsymbol{x}), \quad \boldsymbol{p}^k(\boldsymbol{x}, 0) = \boldsymbol{p}_k^0(\boldsymbol{x}), \quad y^k(\boldsymbol{x}, 0) = y_k^0(\boldsymbol{x}),
\end{aligned} \tag{5.21}$$

where

$$\mathscr{K} := \{(\boldsymbol{\sigma}, y) \in M_s^{3\times 3} \times \mathbb{R} : |\boldsymbol{\sigma}^{\mathrm{D}}| \leqslant c_0 - \eta y\}. \tag{5.22}$$

The free energy corresponding to Problem (5.21) is

$$\rho\psi^k(\boldsymbol{\epsilon}^k, \boldsymbol{p}^k, y^k) := \frac{1}{2}\mathscr{C}(\boldsymbol{\epsilon}^k - \boldsymbol{p}^k) : (\boldsymbol{\epsilon}^k - \boldsymbol{p}^k) + \frac{1}{2}|y^k|^2 + \frac{1}{2k}\mathscr{C}\boldsymbol{\epsilon}^k : \boldsymbol{\epsilon}^k. \tag{5.23}$$

To obtain an existence result for the problem (5.21), the indicator function is regularized using the Yosida, or Perzyna, approximation, with parameter μ as before. Subject to the data satisfying some regularity and compatibility conditions, a unique solution to this problem is obtained. The remainder of the analysis entails repeated application of the strategy of obtaining a priori estimates and extracting convergent subsequences. In this way it is shown that a unique solution to the dynamic problem exists.

The solution $(\boldsymbol{v}, \boldsymbol{\epsilon}, \boldsymbol{p})$ of Problem (5.17) is obtained in the limit, provided that the boundary data $\boldsymbol{g}$ satisfies a safe load condition, and it can be shown that

$$\begin{aligned}
&\boldsymbol{v} \in W^{1,\infty}((0, T); \boldsymbol{L}^2(\Omega)),\\
&y \in W^{1,\infty}((0, T); L^2(\Omega)),\\
&\boldsymbol{\epsilon} \in W_{w^*}^{1,\infty}((0, T); \mathscr{M}(\Omega)),\\
&\boldsymbol{p} \in W_{w^*}^{1,\infty}((0, T); \mathscr{M}(\Omega)),\\
&\boldsymbol{\epsilon} - \boldsymbol{p} \in W^{1,\infty}((0, T); \boldsymbol{L}^2(\Omega)).
\end{aligned}$$

Here $W_{w^*}^{1,\infty}((0, T); \mathscr{M}(\Omega))$ is related to the space $L_{w^*}^{\infty}((0, T); \mathscr{M}(\Omega))$ of essentially bounded and weakly $*$ measurable functions (see Appendix A).

6. A direct approach in the BD and measures setting

The direct approach to studying the quasi-static primal problem for perfect plasticity consists in considering Problem (5.7) in the space $H^1(0,T;Z)$ without expanding the bilinear form $a(\cdot,\cdot)$. We recall that terms like $\boldsymbol{\epsilon}(\boldsymbol{u}):\boldsymbol{\epsilon}(\boldsymbol{v})$ and $\boldsymbol{\epsilon}(\boldsymbol{u}):\boldsymbol{p}$ do not make sense in Z.

Such an approach appears to hold promise, and full details are reported in the work [7]. Starting with the dissipation function D corresponding to the von Mises yield criterion we write the flow law (2.20) in the form

$$\boldsymbol{\sigma}^{\mathrm{D}} = c_0 \frac{\dot{\boldsymbol{p}}}{|\dot{\boldsymbol{p}}|}. \tag{6.1}$$

The main idea of the direct approach is to obtain the solution of Problems (2.1)–(2.13) and (2.20) as a weak limit of an interpolated solution constructed from a finite number of problems obtained through time discretizaton.

To this end the time interval $[0,T]$ is subdivided into N subintervals of equal length, so that $0 = t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = T$ with $t_n := nT/N$, $n = 0,\ldots,N$. For $\ell \in H^1(0,T;Z^*)$ we write $\ell_n = \ell(t_n)$, which is well-defined by the embedding $H^1(0,T;X) \hookrightarrow C(0,T;X)$ for any Banach space X. Denote by $\Delta\boldsymbol{w}_n$ the backward difference $\boldsymbol{w}_n - \boldsymbol{w}_{n-1}$ corresponding to the sequence $\{\boldsymbol{w}_n\}_{n=0}^N$.

Analogously to the static problem for the Hencky law (see for instance [2,35]), we formally write the discretized constitutive law in a way that will be useful later; that is, for an arbitrary step n,

$$\boldsymbol{\sigma}_n^{\mathrm{D}} = c_0 \frac{\Delta\boldsymbol{p}_n}{|\Delta\boldsymbol{p}_n|}, \tag{6.2}$$

with

$$\boldsymbol{\sigma}_n^{\mathrm{D}} := 2\mu(\boldsymbol{\epsilon}^{\mathrm{D}}(\boldsymbol{u}_n) - \boldsymbol{p}_n) = 2\mu(\boldsymbol{\epsilon}^{\mathrm{D}}(\boldsymbol{u}_n) - \Delta\boldsymbol{p}_n - \boldsymbol{p}_{n-1}) = 2\mu(\boldsymbol{\gamma}_n - \Delta\boldsymbol{p}_n).$$

Here $\boldsymbol{\gamma}_n := \boldsymbol{\epsilon}^{\mathrm{D}}(\boldsymbol{u}_n) - \boldsymbol{p}_{n-1}$. So substituting $\boldsymbol{\sigma}_n^{\mathrm{D}}$ in (6.2) we obtain

$$\Delta\boldsymbol{p}_n\left[\frac{c_0}{|\Delta\boldsymbol{p}_n|} + 2\mu\right] = 2\mu\boldsymbol{\gamma}_n,$$

from which we get

$$|\Delta\boldsymbol{p}_n| = |\boldsymbol{\gamma}_n| - c_0/2\mu,$$

and hence

$$\Delta\boldsymbol{p}_n = \boldsymbol{\gamma}_n - \frac{c_0}{2\mu}\frac{\boldsymbol{\gamma}_n}{|\boldsymbol{\gamma}_n|}.$$

Therefore the constitutive relation (6.2) written in terms of the new variable $\boldsymbol{\gamma}_n$ is given by

$$\boldsymbol{\sigma}_n^{\mathrm{D}} = c_0 \frac{\boldsymbol{\gamma}_n}{|\boldsymbol{\gamma}_n|}. \tag{6.3}$$

Taking into account also the elastic part we get

$$\boldsymbol{\sigma}_n^{\mathrm{D}} = \begin{cases} \mu|\boldsymbol{\gamma}_n|^2 & \text{if } |\boldsymbol{\gamma}_n| \leqslant c_0/2\mu, \\ c_0 \dfrac{\boldsymbol{\gamma}_n}{|\boldsymbol{\gamma}_n|} & \text{if } |\boldsymbol{\gamma}_n| \geqslant c_0/2\mu. \end{cases} \tag{6.4}$$

Following the approach by Anzellotti and Giaquinta [2] we consider the following variational problem: for every integer $n \geqslant 1$

$$\min_{\mathbf{v}\in P_0(\Omega)} E_n(\mathbf{v}), \tag{6.5}$$

where $P_0(\Omega) := \{\mathbf{v} \in \mathrm{BD}(\Omega) : \operatorname{div} \mathbf{v} \in L^2(\Omega) \text{ and } \mathbf{v} = 0 \text{ on } \partial\Omega\}$,

$$E_n(\mathbf{v}) := \int_\Omega \Phi(\boldsymbol{\epsilon}^{\mathrm{D}}(\mathbf{v}) - \mathbf{p}_{n-1}) + \frac{\kappa}{2}\int_\Omega (\operatorname{div} \mathbf{v})^2\,\mathrm{d}\mathbf{x} - \int_\Omega \mathbf{f}_n \mathbf{v}\,\mathrm{d}\mathbf{x}, \tag{6.6}$$

and denoting by $\mathbf{M}$ the finite-dimensional space of $d\times d$ symmetric matrices, $\Phi : \mathbf{M} \to \mathbb{R}$ is defined by $\Phi(\boldsymbol{\xi}) := \beta(|\boldsymbol{\xi}|)$ with

$$\beta(s) := \begin{cases} \mu s^2 & \text{if } 0\leqslant s\leqslant c_0/2\mu, \\ c_0 s - c_0^2/4\mu & \text{otherwise}, \end{cases} \tag{6.7}$$

and $\kappa = \lambda + \frac{2}{3}\mu$ is the bulk modulus. We recall that $\Phi(\boldsymbol{\epsilon}^{\mathrm{D}}(\mathbf{v}) - \mathbf{p}_{n-1})$ is defined in the context of convex functions of measures (see Appendix A).

Analogously to the minimal area problem, the boundary condition is relaxed in the following way:

let Ω' be a smooth bounded connected open subset of $\mathbb{R}^3$ such that $\overline{\Omega} \subset \Omega'$. Using the extension by zero in $\Omega' \setminus \overline{\Omega}$ of functions in $P_0(\Omega)$, one can see that the problem (6.5) is equivalent to

$$\min_{\mathbf{v}\in \mathscr{P}_0(\Omega')} \mathscr{E}(\mathbf{v}) \tag{6.8}$$

where $\mathscr{P}_0(\Omega') := \{\mathbf{v} \in P(\Omega') : \mathbf{v} = 0 \text{ in } \Omega' \setminus \overline{\Omega}\}$ and

$$\mathscr{E}(\mathbf{v}) := \int_{\Omega'} \Phi(\boldsymbol{\epsilon}^{\mathrm{D}}(\mathbf{v}) - \mathbf{p}_{n-1}) + \frac{\kappa}{2}\int_{\Omega'} (\operatorname{div} \mathbf{v})^2\,\mathrm{d}\mathbf{x} - \int_{\Omega'} \mathbf{f}_n \cdot \mathbf{v}\,\mathrm{d}\mathbf{x} \tag{6.9}$$

with $\mathbf{f}_n$ extended by zero in $\Omega' \setminus \overline{\Omega}$. Also Problem (6.8) is equivalent to

$$\min_{\mathbf{v}\in P_v(\Omega)} \mathscr{F}_n(\mathbf{v}) \tag{6.10}$$

where

$$\mathscr{F}_n(\underline{\mathbf{n}}) := \int_\Omega \Phi(\boldsymbol{\epsilon}^{\mathrm{D}}(\underline{\mathbf{n}}) - \mathbf{p}_{n-1}) + \frac{\chi}{2}\int_\Omega (\operatorname{div} \underline{\mathbf{n}})^2\,\mathrm{d}\mathbf{x} + c_0\int_\Gamma |\operatorname{sym}(\underline{\mathbf{n}} \otimes \underline{\mathbf{n}})\delta_\Gamma - \mathbf{p}_{n-1}| - \int_\Omega \mathbf{f}_n \cdot \underline{\mathbf{n}}\,\mathrm{d}\mathbf{x}, \tag{6.11}$$

where δ_Γ is the Dirac delta surface measure on Γ (see also (3.5)) and $\underline{\mathbf{n}}$ is the outer unit normal to Γ.

Here $P_{\underline{\mathbf{n}}}(\Omega) : = \{\underline{\mathbf{n}} \in P(\Omega) : \underline{\mathbf{n}} \cdot \underline{\mathbf{n}} = 0 \text{ on } \Gamma\}$.

We will confine ourselves only to external loads of a particular type; in particular, assume that

$$\mathbf{f}(\cdot,t) = \mathbf{h}(\cdot,t) + \nabla q(\cdot,t) \quad \forall t \in [0,T]$$

with

(i) $\mathbf{h} \in H^1(0,T;L^3(\Omega;\mathbb{R}^3))$,
(ii) $q \in H^1(0,T;H^1(\Omega))$, with $q = 0$ on $\Gamma \times (0,T)$,
(iii) $\|\mathbf{h}\|_{H^1(0,T;L^3(\Omega;\mathbb{R}^3))} + \|q\|_{H^1(0,T;L^2(\Omega))} \leqslant \delta$ for some δ sufficiently small.

Note that by the embedding $H^1(0,T;X) \hookrightarrow C(0,T;X)$, for any Banach space X, (iii) implies that

$$\sup_{t\in[0,T]} \left(\|\mathbf{h}(\cdot,t)\|_{L^3(\Omega;\mathbb{R}^3)} + \|q(\cdot,t)\|_{L^2(\Omega;\mathbb{R}^3)}\right) \leqslant C(T)\delta. \tag{6.12}$$

By the direct method of the calculus of variations we get, arguing as in [2], that Problem (6.8) has a solution, and by equivalence, Problem (6.10) also has a solution, denoted by $\mathbf{u}_n$.

Now from the function $\boldsymbol{u}_n$ we define $\boldsymbol{\sigma}_n$ and $\boldsymbol{p}_n$. We recall that

$$\mathscr{F}_n(\boldsymbol{v}) := \int_\Omega \Phi(\boldsymbol{\epsilon}^{\mathrm{D}}(\boldsymbol{v})^{\mathrm{a}} - \boldsymbol{p}^{\mathrm{a}}_{n-1})\,\mathrm{d}\boldsymbol{x} + c_0 \int_\Omega d|\boldsymbol{\epsilon}^{\mathrm{D}}(\boldsymbol{v})^{\mathrm{s}} - \boldsymbol{p}^{\mathrm{s}}_{n-1}| + \frac{\chi}{2}\int_\Omega (\mathrm{div}\,\boldsymbol{v})^2\,\mathrm{d}\boldsymbol{x}$$
$$+ c_0 \int_\Gamma |\mathrm{sym}(\underline{\mathbf{n}} \otimes \underline{\mathbf{n}})\delta_\Gamma - \boldsymbol{p}_{n-1}| - \int_\Omega \boldsymbol{f}_n \cdot \boldsymbol{v}\,\mathrm{d}\boldsymbol{x},$$

where we recall that for any Radon measure α in Ω, the notations α^{a} and α^{s} denote respectively the density of the absolutely continuous part and the singular part of α with respect to the Lebesgue measure.

Now we consider the weak Euler–Lagrange equation for Problem (6.10) using test functions $\boldsymbol{\varphi} \in \boldsymbol{C}^1_0(\Omega)$, that is,

$$\frac{\mathrm{d}\mathscr{F}_n(\boldsymbol{u}_n + t\boldsymbol{\varphi})}{\mathrm{d}t}\Big|_{t=0} = 0.$$

This gives

$$\int_\Omega \beta'(|\boldsymbol{\epsilon}^{\mathrm{D}}(\boldsymbol{u}_n)^{\mathrm{a}} - \boldsymbol{p}^{\mathrm{a}}_{n-1}|)\frac{\boldsymbol{\epsilon}^{\mathrm{D}}(\boldsymbol{u}_n)^{\mathrm{a}} - \boldsymbol{p}^{\mathrm{a}}_{n-1}}{|\boldsymbol{\epsilon}^{\mathrm{D}}(\boldsymbol{u}_n)^{\mathrm{a}} - \boldsymbol{p}^{\mathrm{a}}_{n-1}|} : \boldsymbol{\epsilon}(\boldsymbol{\varphi})\,\mathrm{d}\boldsymbol{x} + \kappa\int_\Omega \mathrm{div}\,\boldsymbol{u}_n\,\mathrm{div}\,\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} - \int_\Omega \boldsymbol{f}_n\cdot\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} = 0. \tag{6.13}$$

We set

$$\begin{cases} \boldsymbol{\sigma}^{\mathrm{D}}_n := \beta'(|\boldsymbol{\epsilon}^{\mathrm{D}}(\boldsymbol{u}_n)^{\mathrm{a}} - \boldsymbol{p}^{\mathrm{a}}_{n-1}|)\dfrac{\boldsymbol{\epsilon}^{\mathrm{D}}(\boldsymbol{u}_n)^{\mathrm{a}} - \boldsymbol{p}^{\mathrm{a}}_{n-1}}{|\boldsymbol{\epsilon}^{\mathrm{D}}(\boldsymbol{u}_n)^{\mathrm{a}} - \boldsymbol{p}^{\mathrm{a}}_{n-1}|}, \\ \boldsymbol{\sigma}^{\mathrm{S}}_n := 3\kappa\,\mathrm{div}\,\boldsymbol{u}_n \end{cases} \tag{6.14}$$

$$\boldsymbol{p}^{\mathrm{a}}_n := \boldsymbol{\epsilon}^{\mathrm{D}}(\boldsymbol{u}_n)^{\mathrm{a}} - \frac{1}{2\mu}\boldsymbol{\sigma}^{\mathrm{D}}_n \quad \text{and} \quad \boldsymbol{p}^{\mathrm{s}}_n := \boldsymbol{\epsilon}^{\mathrm{D}}(\boldsymbol{u}_n)^{\mathrm{s}}. \tag{6.15}$$

Integrating by parts in (6.13), we obtain the equilibrium equation

$$\mathrm{div}\,\boldsymbol{\sigma}_n + \boldsymbol{f}_n = 0.$$

From the choices of $\boldsymbol{\sigma}_n$ and $\boldsymbol{p}_n$ in (6.14) we obtain the discretized constitutive law in this form

$$\boldsymbol{\sigma}^{\mathrm{D}}_n = c_0\frac{\Delta\boldsymbol{p}^{\mathrm{a}}_n}{|\Delta\boldsymbol{p}^{\mathrm{a}}_n|} \quad \text{a.e. in } \Omega. \tag{6.16}$$

The next step is to construct from $(\boldsymbol{u}_n, \boldsymbol{\sigma}_n, \boldsymbol{p}_n)$ the interpolates (piecewise linear or piecewise constant), and then to study their convergence in suitable topologies. The requisite uniform estimates for $\boldsymbol{u}_n$, $\boldsymbol{\sigma}_n$, $\boldsymbol{p}_n$ are summarized in the following lemma.

Lemma 6.1. *There exists a constant $C > 0$ independent of n such that*

$$\|\boldsymbol{u}_n\|_{P(\Omega)} \leqslant C, \tag{6.17}$$

$$\|\boldsymbol{\sigma}_n\|_{L^2(\Omega,\boldsymbol{M})} \leqslant C, \tag{6.18}$$

$$|\boldsymbol{p}_n|(\Omega) \leqslant C. \tag{6.19}$$

Proof. Note that by construction (6.18) and (6.19) easily follow from (6.17), which is derived in [7]. □

Let us now construct the interpolates. We set

$$\overline{\boldsymbol{u}}_N(t) := \boldsymbol{u}_{n-1} + \frac{t - t_{n-1}}{t_n - t_{n-1}}(\boldsymbol{u}_n - \boldsymbol{u}_{n-1}) \quad \text{when } t \in [t_{n-1}, t_n], \tag{6.20}$$

$$\overline{\boldsymbol{\sigma}}_N(t) := \boldsymbol{\sigma}_n^{\mathrm{D}} + \kappa \operatorname{div} \overline{\boldsymbol{u}}_N(t)\boldsymbol{I} \quad \text{when } t \in [t_{n-1}, t_n], \tag{6.21}$$

$$\begin{aligned}(\overline{\boldsymbol{p}}_N)^{\mathrm{a}}(t) &:= (\boldsymbol{\epsilon}^{\mathrm{D}}(\overline{\boldsymbol{u}}_N))^{\mathrm{a}}(t) - \frac{\overline{\boldsymbol{\sigma}}_N^{\mathrm{D}}(t)}{2\mu} \quad \text{when } t \in [t_{n-1}, t_n],\\ &= (\boldsymbol{\epsilon}^{\mathrm{D}}(\boldsymbol{u}_{n-1}))^{\mathrm{a}} + \frac{t - t_{n-1}}{t_n - t_{n-1}}(\boldsymbol{\epsilon}^{\mathrm{D}}(\boldsymbol{u}_n))^{\mathrm{a}} - (\boldsymbol{\epsilon}^{\mathrm{D}}(\boldsymbol{u}_{n-1}))^{\mathrm{a}} - \frac{\overline{\boldsymbol{\sigma}}_N^{\mathrm{D}}(t)}{2\mu} \quad \text{when } ft \in [t_{n-1}, t_n].\end{aligned} \tag{6.22}$$

Lemma 6.2. *There exists a constant C, independent of N, such that the following hold*:

$$\|\overline{\boldsymbol{u}}_N\|_{L^\infty(0,T;\mathrm{BD}(\Omega))} \leqslant C, \tag{6.23}$$

$$\|\mathrm{div}_x \overline{\boldsymbol{u}}_N\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C, \tag{6.24}$$

$$\|\overline{\boldsymbol{\sigma}}_N^{\mathrm{D}}\|_{L^\infty(0,T;L^\infty(\Omega,\boldsymbol{M}))} \leqslant C, \tag{6.25}$$

$$\|\overline{\boldsymbol{\sigma}}_N\|_{L^\infty(0,T;L^2(\Omega,\boldsymbol{M}))} \leqslant C, \tag{6.26}$$

$$\|\overline{\boldsymbol{p}}_N\|_{L^\infty(0,T;\mathscr{M}(\Omega,\boldsymbol{M}_0))} \leqslant C. \tag{6.27}$$

These estimates follow easily from the estimates of $(\boldsymbol{u}_n, \boldsymbol{\sigma}_n, \boldsymbol{p}_n)$ in Lemma 6.1.

Lemma 6.3. *We have*, *up to a subsequence*,

$$\overline{\boldsymbol{u}}_N \overset{*}{\rightharpoonup} \boldsymbol{u} \quad \text{in } L^\infty_{w^*}(0, T; \mathrm{BD}(\Omega)),$$

i.e.,

$$\lim_{N\to\infty} \int_0^T \int_\Omega \overline{\boldsymbol{u}}_N(t,\boldsymbol{x}) \cdot \boldsymbol{w}(t,\boldsymbol{x})\phi(t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \int_0^T \int_\Omega \boldsymbol{u}(t,\boldsymbol{x}) \cdot \boldsymbol{w}(t,\boldsymbol{x})\phi(t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \quad \forall(\phi, \boldsymbol{w}) \in L^1(0,T) \times \mathscr{C}_0(\Omega, \mathbb{R}^3),$$

and

$$\lim_{N\to\infty} \int_0^T \phi(t)\langle \boldsymbol{\epsilon}(\overline{\boldsymbol{u}}_N), \boldsymbol{\tau}\rangle\,\mathrm{d}t = \int_0^T \phi(t)\langle \boldsymbol{\epsilon}(\boldsymbol{u}), \boldsymbol{\tau}\rangle\,\mathrm{d}t \quad \forall(\phi, \boldsymbol{\tau}) \in L^1(0,T) \times \mathscr{C}_0(\Omega, \boldsymbol{M}).$$

$$\operatorname{div} \overline{\boldsymbol{u}}_N \overset{*}{\rightharpoonup} \operatorname{div} \boldsymbol{u} \quad \text{in } L^\infty_{w^*}(0, T; L^2(\Omega)),$$

i.e.,

$$\lim_{N\to\infty} \int_0^T \int_\Omega \operatorname{div} \overline{u}_N(t,\boldsymbol{x}) w(\boldsymbol{x})\phi(t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \int_0^T \int_\Omega \operatorname{div} \boldsymbol{u}(t,\boldsymbol{x}) w(\boldsymbol{x})\phi(t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \quad \forall(\phi, w) \in L^1(0,T) \times L^2(\Omega).$$

$$\overline{\boldsymbol{\sigma}}_N^{\mathrm{D}} \overset{*}{\rightharpoonup} \boldsymbol{\sigma}^{\mathrm{D}} \quad \text{in } L^\infty_{w^*}(0, T; \boldsymbol{L}^\infty(\Omega)),$$

i.e.,

$$\lim_{N\to\infty} \int_0^T \int_\Omega \overline{\boldsymbol{\sigma}}_N^{\mathrm{D}}(t,\boldsymbol{x}) : \boldsymbol{\tau}(\boldsymbol{x})\phi(t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \int_0^T \int_\Omega \boldsymbol{\sigma}^{\mathrm{D}}(t,\boldsymbol{x}) : \boldsymbol{\tau}(\boldsymbol{x})\phi(t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \quad \forall(\phi, \boldsymbol{w}) \in L^1(0,T) \times \boldsymbol{L}^1(\Omega).$$

$$\overline{\boldsymbol{\sigma}}_N \overset{*}{\rightharpoonup} \boldsymbol{\sigma} \quad \text{in } L^\infty_{w^*}(0, T; \boldsymbol{L}^2(\Omega)),$$

i.e.,

$$\lim_{N\to\infty}\int_0^T\int_\Omega \overline{\boldsymbol{\sigma}}_N(t,\boldsymbol{x}):\boldsymbol{\tau}(\boldsymbol{x})\phi(t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t=\int_0^T\int_\Omega \boldsymbol{\sigma}(t,\boldsymbol{x}):\boldsymbol{\tau}(\boldsymbol{x})\phi(t)\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \quad \forall(\phi,\boldsymbol{\tau})\in L^1(0,T)\times \boldsymbol{L}^2(\Omega).$$

$$\overline{\boldsymbol{p}}_N \overset{*}{\rightharpoonup} \boldsymbol{p} \quad \textit{in } L^\infty_{w^*}(0,T;\mathscr{M}(\Omega)),$$

i.e.,

$$\lim_{N\to\infty}\int_0^T \phi(t)\langle\overline{\boldsymbol{p}}_N,\boldsymbol{\tau}\rangle\,\mathrm{d}t=\int_0^T \phi(t)\langle\boldsymbol{p},\boldsymbol{\tau}\rangle\,\mathrm{d}t \quad \forall(\phi,\boldsymbol{w})\in L^1(0,T)\times\mathscr{C}_0(\Omega,\boldsymbol{M}).$$

So we get a triple

$$(\boldsymbol{u},\boldsymbol{\sigma},\boldsymbol{p})\in L^\infty_{w^*}(0,T;\mathrm{BD}(\Omega))\times L^\infty_{w^*}(0,T;\boldsymbol{L}^2(\Omega))\times L^\infty_{w^*}(0,T;\mathscr{M}(\Omega))$$

with

$$\mathrm{div}\,\boldsymbol{u}\in L^\infty_{w^*}(0,T;L^2(\Omega)),$$

and the final step is to prove that it solves in a suitable weak sense the quasi-static primal problem in perfect plasticity. More precisely, we have to show that the triple $(\boldsymbol{u},\boldsymbol{\sigma},\boldsymbol{p})$ satisfies

$$\begin{cases}\mathrm{div}\,\boldsymbol{\sigma}+\boldsymbol{f}=0 & \text{in } (0,T)\times\Omega,\\ \boldsymbol{\sigma}=\kappa(\mathrm{div}\,\boldsymbol{u})\boldsymbol{I}+2\mu(\boldsymbol{\epsilon}^{\mathrm{D}}(\boldsymbol{u})-\boldsymbol{p}) & \\ \boldsymbol{u}\cdot\underline{\mathrm{n}}=0 & \text{on } (0,T)\times\partial\Omega,\\ \boldsymbol{\sigma}^{\mathrm{D}}=c_0\dfrac{\dot{\boldsymbol{p}}^{\mathrm{a}}}{|\dot{\boldsymbol{p}}^{\mathrm{a}}|} & \text{a.e. in } (0,T)\times\Omega.\end{cases} \tag{6.28}$$

Since the first three equations in (6.28) are linear relations among the variables, one may expect these to hold in the usual weak sense from the properties of weak $*$ convergence established in Lemma 6.3. That is, we expect to have, for instance,

$$\int_0^T\int_\Omega \boldsymbol{\sigma}(t,\boldsymbol{x}):\nabla\boldsymbol{w}(\boldsymbol{x})\phi(t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t=\int_0^T\int_\Omega \boldsymbol{f}(t,\boldsymbol{x})\boldsymbol{w}(\boldsymbol{x})\phi(t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \quad \forall(\phi,\boldsymbol{w})\in L^1(0,T)\times C^1_c(\Omega,\mathbb{R}^3),$$

$$\int_\Omega \boldsymbol{\sigma}(t,\boldsymbol{x}):\boldsymbol{\tau}(\boldsymbol{x})\phi(t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t=\kappa\int_0^T\int_\Omega \mathrm{div}\,\boldsymbol{u}(t,\boldsymbol{x})\mathrm{tr}\,\boldsymbol{\tau}(\boldsymbol{x})\phi(t)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t+2\mu\int_0^T\phi(t)\langle\boldsymbol{\epsilon}^{\mathrm{D}}(\boldsymbol{u})-\boldsymbol{p},\boldsymbol{\tau}\rangle\,\mathrm{d}t$$
$$\forall(\phi,\boldsymbol{\tau})\in L^1(0,T)\times\mathscr{C}_0(\Omega,\boldsymbol{M}),$$

which give the first two equations in (6.28).

The third relation $\boldsymbol{u}\cdot\underline{\mathrm{n}}=0$ on $(0,T)\times\partial\Omega$ holds in the sense of normal trace on $\partial\Omega$ of vector fields (see [35, Proposition 7.2]) for almost every fixed $t\in(0,T)$.

Finally, under additional assumptions on $\dot{\boldsymbol{f}}$, the nonlinear relation $\boldsymbol{\sigma}^{\mathrm{D}}=c_0\dot{\boldsymbol{p}}^{\mathrm{a}}/|\dot{\boldsymbol{p}}^{\mathrm{a}}|$ will hold in the form

$$\boldsymbol{\sigma}^{\mathrm{D}}\in\partial D(\dot{\boldsymbol{p}}^{\mathrm{a}}) \quad \text{for a.e. } (t,\boldsymbol{x})\in(0,T)\times\Omega, \tag{6.29}$$

where D is the dissipation function in (2.24). This problem is currently receiving attention (see [7]).

The main difficulty in our approach is to obtain the flow law (6.29). This is due to the presence of time derivative. Recently a derivative-free energetic formulation for the inelastic behavior of "standard generalized materials" has been proposed by Mielke in [24], with application to linearized elasto-plasticity with hardening. It would be interesting to see whether this formulation can be extended to the case of perfect plasticity.

Appendix A

We summarize in this appendix various notations, definitions, and results from functional analysis and function spaces that are relevant to the analyses presented in the main part of the paper. We will make use of the space $L^2(\Omega)$ of square-integrable functions defined on Ω, with inner product and norm being denoted by $(\cdot,\cdot)_0$ and $\|\cdot\|_0$, respectively. We recall also the definition of the Sobolev spaces $H^m(\Omega)$, with m being a non-negative integer, as equivalence classes of functions with generalized derivatives of order $\leqslant m$ in $L^2(\Omega)$. The Sobolev spaces are Hilbert spaces with inner product and associated norm.

$$(u,v)_m := \int_\Omega \sum_{|\boldsymbol{\alpha}|\leqslant m} D^{\boldsymbol{\alpha}} u(\boldsymbol{x}) D^{\boldsymbol{\alpha}} v(\boldsymbol{x})\,\mathrm{d}x \quad \text{and} \quad \|v\|_m := (v,v)_m^{1/2}. \tag{A.1}$$

Here $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_d)$ is a multi-index whose components α_j are nonnegative integers, $|\boldsymbol{\alpha}| := \alpha_1 + \cdots + \alpha_d$, and as usual $D^{\boldsymbol{\alpha}} = \partial^{|\boldsymbol{\alpha}|}/\partial x_1^{\alpha_1}\cdots\partial x_d^{\alpha_d}$.

The semi-norm $|\cdot|_m$ on $H^m(\Omega)$ is defined by

$$|v|_m^2 := \int_\Omega \sum_{|\boldsymbol{\alpha}|=m} D^{\boldsymbol{\alpha}} v(\boldsymbol{x}) D^{\boldsymbol{\alpha}} v(\boldsymbol{x})\,\mathrm{d}x. \tag{A.2}$$

The space $H_0^1(\Omega)$ consists of functions in $H^1(\Omega)$ which vanish on the boundary in the sense of traces. The space $H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$.

The finite-dimensional space of $d\times d$ symmetric matrices is denoted by $\boldsymbol{M}$. Vector- and matrix-valued function spaces will be denoted in the same way as their real-valued counterparts, but using bold-faced letters. Thus the space of vector- or symmetric matrix-valued square-integrable functions will be denoted by $\boldsymbol{L}^2(\Omega)$, with inner product and norm generated in the usual way from the real-valued case.

Radon measures (see Temam [35]). Let Ω be a bounded open subset of $\mathbb{R}^n$. We denote by $\mathscr{C}_0(\Omega)$ the space of continuous functions on Ω vanishing at the boundary. We recall that this space is the completion with respect to the uniform norm of the space $\mathscr{C}_c(\Omega)$ having compact support in Ω. A Radon measure is a linear continuous functional on the space $\mathscr{C}_0(\Omega)$. Note that from Hahn–Banach extension theorem it is not restrictive to define a Radon measure as a linear continuous (with respect to the supremum norm) functional on the space $\mathscr{C}_c(\Omega)$.

Given a Radon measure μ, we define the total variation of μ, denoted by $|\mu|(\Omega)$, as

$$|\mu|(\Omega) := \sup_{\phi\in\mathscr{C}_0(\Omega)} \frac{|\langle\mu,\phi\rangle|}{\|\phi\|_\infty} \tag{A.3}$$

where $\|\cdot\|_\infty$ is the supremum norm.

We say that a Radon measure μ is bounded when its total variation $|\mu|(\Omega)$ is finite. The space of bounded Radon measures on Ω is denoted by $\mathscr{M}(\Omega)$; this is a Banach space when equipped with the norm

$$\|\mu\|_{\mathscr{M}} := |\mu|(\Omega).$$

For any $f\in L^1(\Omega)$ one may define a bounded linear functional F on $C(\overline{\Omega})$ according to

$$\langle F,\varphi\rangle = \int_\Omega f(\boldsymbol{x})\varphi(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \quad \forall\varphi\in C(\overline{\Omega}).$$

Thus $L^1(\Omega)$ is continuously embedded in $\mathscr{M}(\Omega)$.

As a dual space $\mathscr{M}(\Omega)$ is also equipped with the so-called weak $*$ topology induced by the family of semi-norms

$$|\langle\mu,\phi\rangle| \quad \text{with } \phi\in\mathscr{C}_0(\Omega).$$

So we say that a sequence of measures (μ_n) converges weakly $*$ to a Radon measure μ if

$$\langle \mu_n, \phi \rangle \rightarrow \langle \mu, \phi \rangle \quad \forall \phi \in \mathscr{C}_0(\Omega).$$

We denote by $\mathscr{M}$ the space of symmetric matrix-valued Radon measures with finite total variation (this will typically be the space to which the plastic strains belong).

Convex functions of measures (see Temam [35, Chapter II Section 4]). Let $f : \boldsymbol{M} \rightarrow [0, +\infty[$ be a convex and lower semicontinuous function, with $f(0) = 0$, and $f(\boldsymbol{\xi}) \leqslant c(|\boldsymbol{\xi}| + 1)$ for every $\boldsymbol{\xi} \in \boldsymbol{M}$ and for some positive constant c.

The Fenchel conjugate of f is defined for every $\boldsymbol{\eta} \in \boldsymbol{M}$ by

$$f^*(\boldsymbol{\eta}) = \sup_{\boldsymbol{\xi} \in \boldsymbol{M}} \{ \boldsymbol{\eta} : \boldsymbol{\xi} - f(\boldsymbol{\xi}) \}. \tag{A.4}$$

Since $f(\mathbf{0}) = 0$, the function f^* takes its values in $[0, +\infty]$ and we define its domain (of finiteness) by

$$K = \{ \boldsymbol{\eta} \in \boldsymbol{M} : f^*(\boldsymbol{\eta}) < +\infty \}.$$

Note that K is not empty since f is finite.

We introduce the function f_∞ called the support function of K, defined by

$$f_\infty(\boldsymbol{\xi}) = \sup\{ \boldsymbol{\eta} : \boldsymbol{\xi} \,|\, \boldsymbol{\eta} \in K \}. \tag{A.5}$$

Let $\mu \in \mathscr{M}$. We define, for $\phi \in \mathscr{C}_0(\Omega)$ with $\phi \geqslant 0$

$$\langle f(\mu), \phi \rangle = \sup_{\boldsymbol{v} \in \mathscr{D}_f(\mathscr{C}_0)} \left\{ \int_\Omega \phi \boldsymbol{v} \, \mathrm{d}\mu - \int_\Omega f^*(\boldsymbol{v}) \phi \, \mathrm{d}\boldsymbol{x} \right\}, \tag{A.6}$$

where f^* is the Fenchel-conjugate of the function f defined in (A.4) and

$$\mathscr{D}_f(\mathscr{C}_0) = \{ \boldsymbol{v} \in \mathscr{C}_0(\Omega, \boldsymbol{M}) : f^* \circ \boldsymbol{v} \in L^1(\Omega) \}.$$

It can be shown that $f(\mu)$ is a positive Radon measure absolutely continuous with respect to μ. Moreover, from the Lebesgue decomposition of μ,

$$\mu = h \, \mathrm{d}x + \mu^{\mathrm{s}}$$

with $h \in L^1(\Omega, \boldsymbol{M})$ and μ^{s} singular with respect to dx, and we have

$$f(\mu) = (f \circ h) \mathrm{d}x + f_\infty(\mu^{\mathrm{s}}),$$

where f_∞ is defined in (A.5).

For more details, we refer the reader to Temam [35, Theorem 4.2].

The following theorem by Reshetniak [30] is used to prove the existence of the solution for the discretized problems in Section 5.

Theorem A.1. *Let* $f : \mathbb{R}^m \rightarrow [0, +\infty]$ *be a convex and lower semicontinuous function such that* $0 \leqslant f(\xi) \leqslant c(|\xi|+1)$, *then the functional* $\mu \rightarrow \int_\Omega f(\mu)$ *is sequentially weakly $*$ lower semicontinuous on* $\mathscr{M}(\Omega)$, *i.e.*, $\int_\Omega f(\mu) \leqslant \liminf_{n \rightarrow \infty} \int_\Omega f(\mu_h)$ *for* (μ_n) *weakly $*$ converging to the measure* μ.

The space BD(Ω). The space of functions of bounded deformation is defined by

$$\mathrm{BD}(\Omega) = \{ \boldsymbol{v} \in [L^1(\Omega)]^d \mid \boldsymbol{\epsilon}(\boldsymbol{v}) \in \mathscr{M} \}. \tag{A.7}$$

Thus BD(Ω) consists of displacements that are integrable, and the corresponding strains of which are bounded measures.

The space BD(Ω) is a Banach space with norm $\|\cdot\|_{\mathrm{BD}}$ defined by

$$\|\boldsymbol{v}\|_{\mathrm{BD}} := \|\boldsymbol{v}\|_{L^1} + \sum_{i,j=1}^{d} \|\epsilon_{ij}(\boldsymbol{v})\|_{\mathscr{M}}. \tag{A.8}$$

The space BD(Ω) is continuously embedded in L^p spaces for certain values of p; in particular,

$$\mathrm{BD}(\Omega) \hookrightarrow [L^{d^*}(\Omega)]^d \quad \text{for } d^* = d/(d-1).$$

Furthermore, BD(Ω) is compactly embedded in $\boldsymbol{L}^q(\Omega)$ for all q such that $1 \leqslant q < d^*$. This means that bounded sets in BD(Ω) are mapped to relatively compact sets (see below) in $\boldsymbol{L}^q(\Omega)$.

Compactness properties. A powerful tool in establishing the existence of solutions to nonlinear partial differential equations is one based on properties of compact sets. A subset Y of a normed space X is (sequentially) compact if every bounded sequence in Y contains a convergent subsequence. The set Y is weakly compact if every bounded sequence contains a weakly convergent subsequence, that is, a subsequence $\{y_n\}$ with the property that there exists $y \in Y$ such that

$$\langle \ell, y_n \rangle \to \langle \ell, y \rangle$$

for all bounded linear functionals ℓ on Y.

A large class of weakly compact sets in characterized in the following, known as the Eberlein–Smulyan Theorem.

Theorem A.2. *Every bounded subset of a reflexive Banach space is relatively weakly compact.*

Reflexive normed spaces are those for which the bidual (that is, the dual of the dual) may be identified with the original space. It is a property possessed by all Hilbert spaces.

A related result is the Banach–Alaoglu–Bourbaki Theorem.

Theorem A.3. *Every bounded set in the dual X' of a normed space X is relatively weakly compact.*

It follows from this theorem that every uniformly bounded sequence of measures $(\mu_n) \subset \mathscr{M}(\Omega)$ converges (up to a subsequence) weakly $*$ to a measure $\mu \in \mathscr{M}(\Omega)$.

BD(Ω) is weakly sequentially compact, in the sense that every bounded sequence $\{\boldsymbol{u}_m\}$ in BD(Ω) contains a subsequence, again denoted by $\{\boldsymbol{u}_m\}$, such that

$$\begin{aligned} &\boldsymbol{u}_m \to \boldsymbol{u} \quad \text{in } [L^1(\Omega)]^d, \\ &\epsilon_{ij}(\boldsymbol{u}_m) \to \epsilon_{ij}(\boldsymbol{u}) \quad \text{weakly in } \mathscr{M}(\Omega). \end{aligned} \tag{A.9}$$

Strongly and weakly measurable functions. Let X be a Banach space with topological dual X' and T be a positive real number.

A function $f: (0,T) \to X$ is said to be *strongly measurable* if there exists a sequence of simple measurable functions (f_j) such that $f_j(t) \to f(t)$ a.e. $t \in (0,T)$ and

$$\int_0^T \|f_j(t) - f_k(t)\|_X^p \, \mathrm{d}t \to 0 \quad \text{as } j,k \to \infty.$$

A function $f: (0,T) \to X$ (resp. $f: (0,T) \to X'$) is said to be *weakly measurable* (resp. *weakly $*$ measurable*) if for every $x^* \in X'$ (resp. for every $x \in X$) we have that the function $t \in (0,T) \to \langle x^*, f(t) \rangle$ (resp. $t \in (0,T) \to \langle f(t), x \rangle$) is measurable.

For, $1 \leqslant p < \infty$ we set

$$L^p(0,T;X) := \left\{ f : (0,T) \to X \text{ strongly measurable and } \int_0^T \|f(t)\|_X^p \, \mathrm{d}x < \infty \right\},$$

$$L^p_w(0,T;X) := \left\{ f : (0,T) \to X \text{ weakly measurable and } \int_0^T \|f(t)\|_X^p \, \mathrm{d}x < \infty \right\},$$

$$L^p_{w\star}(0,T;X') := \left\{ f : (0,T) \to X' \text{ weakly } * \text{ measurable and } \int_0^T \|f(t)\|^p_{X'}\,\mathrm{d}t < \infty \right\}.$$

For $p = \infty$ we use the essential supremum norm.

If X is a separable Banach space then

$$(L^p(0,T;X))' = L^q_{w\star}(0,T;X') \quad \text{where } q = p/(p-1) \text{ for every } 1 \leqslant p < \infty.$$

For instance

$$L^\infty_{w\star}(0,T;\mathscr{M}(\Omega)) = (L^1(0,T;\mathscr{C}_0(\Omega,\boldsymbol{M})))' \quad \text{and} \quad L^\infty_{w\star}(0,T;\mathrm{BD}(\Omega)) = (L^1(0,T;X(\Omega)))'$$

as the space $\mathrm{BD}(\Omega)$ of functions with bounded deformation is known to be dual of some subspace $X(\Omega)$ of continuous functions while

$$L^\infty_{w\star}(0,T;\boldsymbol{L}^2(\Omega)) = (L^1(0,T;\boldsymbol{L}^2(\Omega))', \quad \text{and} \quad L^\infty_{w\star}(0,T;\boldsymbol{L}^\infty(\Omega)) = (L^1(0,T;\boldsymbol{L}^1(\Omega)))'.$$

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