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Well-posedness of the problem of fiber suspension flows

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Abstract

The well-posedness of the equations governing the flow of fiber suspensions is studied. The fluid is assumed to be Newtonian and incompressible, and the presence of fibers is accounted for through the use of second- and fourth-order orientation tensors, which model the effects of the orientation of fibers in an averaged sense. The fourth-order orientation tensor is expressed in terms of the second-order tensor through various closure relations. It is shown that the linear closure relation leads to anomalous behavior, in that the rest state of the fluid is unstable, in the sense of Liapounov, for certain ranges of the fiber particle number. No such anomalies arise in the case of quadratic and hybrid closure relations. For the quadratic closure relation, it is shown that a unique solution exists locally in time for small data. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

The rheological behaviour of a suspension of rigid axisymmetric particles in a viscous fluid is a subject of great industrial importance, for example, in the manufacture of short-fiber composites. These are generally formed by automated methods such as injection molding or compression molding of fiber suspensions. The presence of fibers affects the flow of the fluid, and this in turn affects the fiber orientation, so that the problem of determining the flow characteristics is highly coupled.

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Constitutive theories for fiber suspensions are at the present time well-developed, at least in the case of non-concentrated suspensions in Newtonian fluids, and good accounts may be found in the works by Dinh and Armstrong [1], Leal and Hinch [2], Lipscomb et al. [3], and Tucker and Advani [4]. Analyses of various special flows—sometimes coupled with discussion of related experimental findings—have been carried out by Alexandrou and Ahmed [5], Evans [6], Jackson et al. [7], Ranganathan and Advani [8], and Tucker [9].

Numerical simulations of the processing of fiber suspensions are particularly relevant to related industrial processes, since such simulations are able to supplement, or in some cases even supplant, costly experimental investigations. A numerical treatment of the problem, together with representative examples, may be found in the work by Rosenberg et al. [10], who present results on the simulation of non-recirculating flows. Reddy and Mitchell [11] also present details of a numerical investigation. Their work focusses on two features: first, the instabilities that can occur with the use of the linear closure approximation (Section 3 of this work), and the behavior of fiber suspension flows in domains which are characterized by abrupt changes in geometry.

The qualititative properties of the equations governing fiber suspension flows have, on the other hand, not been studied, and the issue of well-posedness remains open. This is in contrast to the situation which pertains in respect to non-Newtonian flows (of which fiber suspensions may be regarded as a particular example): for example, the issues of existence and uniqueness of solutions have been investigated for viscoelastic fluids with differential constitutive laws [12–14], and for second grade fluids [15].

The purpose of this contribution is to establish the existence and uniqueness of solutions to the equations governing the flow of fiber suspensions. The particular model studied is one in which the presence of fibers is accounted for by the inclusion of a second-order tensor known as the orientation tensor, which accounts in an averaged sense for the distribution of fibers in the fluid. The use of orientation tensors was first suggested by Hand [16], and the idea has been investigated in some detail by Tucker and Advani [9,17,18], whose approach will form the basis of the model analyzed here.

The use of the orientation tensor as a field variable has the advantage that the behavior of fibers may be characterized in an averaged way, and in a manner which permits a completely deterministic problem to be treated. Thus, there is no need to deal explicitly with the randomness of fiber orientation through probability density functions and the like. A further advantage to be gained is that the orientation tensors are continuum quantities, so that the governing equations, as well as subsequent analyses and computational studies, may be founded on well-known approaches.

The approach taken in this study is to treat the problem of fiber suspension flows as far as possible as an extension of the problem of determining flows of Newtonian fluids. That is, the foundation for the study is the Navier-Stokes equations; these are modified to accommodate the presence of fibers, and they are also supplemented by equations which govern the evolution of orientation of fibers.

The rest of this work is organized as follows. The mechanics of dilute and semi-dilute fiber suspensions are described in Section 2, the orientation tensors are introduced, and the equations governing their behavior are reviewed. The modifications in the constitutive equation for the stress are also described here.

A feature of any approach involving orientation tensors is the necessity of approximating higher-order tensors in terms of those of lower order. This procedure is known as a closure approximation [4]. In the present situation, it will be necessary to carry out this procedure for the fourth-order tensor $\mathscr{A}$, which is approximated as a function of the second-order tensor $\boldsymbol{A}$. Such an approximation can be made in a variety of ways, for example as either a linear or quadratic approximation, or as a hybrid approximation which involves a combination of the linear and quadratic functions. In Section 3 we show that certain anomalous features occur in the case of the linear closure approximation; indeed, it will be shown there that if the particle number $N_{\rm p}$, a key material constant, satisfies $N_{\rm p} > 35/2$, then the rest state is unstable in the sense of Liapounov.

The problem to be studied is that corresponding to the quadratic closure approximation; this problem is posed in Section 4, where it is shown that it can be reduced to the consideration of two suitably linearized problems. In Section 5 the unique solvability of these two auxiliary problems is established. Finally, in Section 6 we prove local (in time) existence for small data, and local uniqueness (for small data) of solutions to the full nonlinear problem. These results could also be obtained, using the same methods, for the hybrid closure approximation, and for the linear approximation for the case $N_{\rm p} \leq 35/2$. It has not been possible to prove the global existence of solutions, at least using conventional methods, and this is left as an open problem.

1.1. Notation

Conventional vector, tensor and indicial notation are used. Coordinate-free notation is used wherever convenient; components are always referred to a fixed orthonormal basis. In addition, the summation convention is invoked in respect of repeated indices.

The letters c, C, K will throughout this work denote positive constants whose numerical value or dependence on parameters is not essential to our aims. In such cases c, C, K may have several different values in a single sequence of manipulations. For example, we may have, in the same line, $2c \leq c$.

2. Fiber suspensions

Consider firstly a typical fiber in a Newtonian fluid. The fiber is assumed to be axisymmetric—either cylindrical or ellipsoidal—with orientation described by the unit vector $\boldsymbol{p}$. A suspension of uniform cylindrical or ellipsoidal fibers is characterized by the particle volume fraction h and the fiber aspect ratio $r = \ell/d$, in which ℓ and d are, respectively, the fiber length and diameter.

A suspension is said to be [4]

$$\left.\begin{array}{c}\text{dilute}\\ \text{semi-dilute}\\ \text{concentrated}\end{array}\right\} \text{ if } \left\{\begin{array}{c} hr^2 < 1,\\ 1 < hr^2 < r,\\ r < hr^2.\end{array}\right. \tag{2.1}$$

We will be concerned with dilute and semi-dilute suspensions, for which fibers have a low probability of making contact, though (in the semi-dilute case) the fiber and fluid motion are coupled.

The orientation of fibers will differ from point to point, as well as in time, and it is unrealistic to attempt a proper description of this variation in orientation. Instead, a more useful approach is to introduce a probability density function $\psi(\boldsymbol{p})$ whose value for a given orientation gives the probability that a fiber has that particular orientation. However, there are practical difficulties in using ψ in realistic simulations, since it would be necessary to carry out computations at a great many points in order to track the evolution and spatial variation of this quantity.

The orientation tensors provide a way around this difficulty. In order to introduce these we first define the averaging operator $\langle\cdot\rangle$ by

$$\langle f\rangle = \int f(\boldsymbol{p})\psi(\boldsymbol{p})\,\mathrm{d}\boldsymbol{p} \equiv \int_{S^1} f(\theta,\phi)\psi(\theta,\phi)\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi; \tag{2.2}$$

here f is an arbitrary function, and integration is over the unit sphere S^1, with respect to spherical coordinates θ and ϕ. Thus, the integration is taken over all possible orientations. We also observe that, in view of the definition of the probability density,

$$\langle 1\rangle = 1.$$

The orientation tensors are obtained by averaging tensor products of the vector $\boldsymbol{p}$. The case of a first-order tensor or vector is trivial, since $\langle\boldsymbol{p}\rangle = 0$ by definition, and the same applies to any tensor of odd order constructed in this way. Therefore, we are left with even-ordered tensors; the second-order orientation tensor $\boldsymbol{A}$ is given by

$$\boldsymbol{A} = \langle\boldsymbol{p}\otimes\boldsymbol{p}\rangle \text{ or } A_{ij} = \langle p_i p_j\rangle, \tag{2.3}$$

and the fourth-order tensor $\mathscr{A}$ by

$$\mathscr{A} = \langle\boldsymbol{p}\otimes\boldsymbol{p}\otimes\boldsymbol{p}\otimes\boldsymbol{p}\rangle \text{ or } \mathscr{A}_{ijkl} = \langle p_i p_j p_k p_l\rangle. \tag{2.4}$$

By virtue of its definition and the properties of ψ, the tensor $\boldsymbol{A}$ satisfies the conditions

$$A_{ji} = A_{ij} \text{ and } A_{ii} = 1, \tag{2.5}$$

so that there are only five independent components in three dimensions and two for plane situations. Likewise, the fourth-order tensor $\mathscr{A}$ has the symmetries

$$\mathscr{A}_{ijkl} = \mathscr{A}_{jikl} = \mathscr{A}_{kijl} = \mathscr{A}_{lijk} = \mathscr{A}_{klij}, \text{ etc.} \tag{2.6}$$

Furthermore, the higher-order tensors give complete information about their lower-order counterparts; for example, from Eq. (2.4) we have

$$\mathscr{A}_{ijkk} = A_{ij}. \tag{2.7}$$

A complete representation of ψ can be given in the form of an infinite series, involving orientation tensors of all orders. It follows therefore that the use of orientation tensors up to a particular order amounts to an approximation. Most models, including those discussed here, make use of the second-order tensor as a measure of orientation, and define the higher-order tensors in an approximate manner, in terms of the second-order tensor.

It may be extremely complicated to represent ψ in all its generality, particularly if the motion is unsteady and nonhomogeneous, in which case $\psi = \psi(\boldsymbol{p}, \boldsymbol{x}, t)$. As a result, various means of circumventing the explicit use of the probability density have been proposed:

2.1. Alignment with velocity [3,6]

In this approach it is assumed that $\boldsymbol{p}$ is parallel to the fluid velocity field $\boldsymbol{v}$.

2.2. Orientation as a function of deformation [1]

Here, it is assumed that

$$\psi(\boldsymbol{p}) = \frac{1}{4\pi}(\boldsymbol{p} \cdot \boldsymbol{B}\boldsymbol{p})^{-3/2}$$

where $\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^{\mathrm{T}}$ is the right Cauchy–Green deformation tensor, $\boldsymbol{F}$ being the deformation gradient. This approximation is valid for infinitely slender fibers with no interaction and random initial orientation. This scheme eliminates the need to determine ψ exactly, but it has limitations, in that it cannot be used for models that contain diffusion or interaction terms.

2.3. Direct calculation of orientation tensors [4,9,17]

This approach, which forms the basis of the model analyzed here, allows the probability function to be eliminated in favor of an evolution equation for $\boldsymbol{A}$. This is achieved by first introducing a conservation equation for fibers, in the form

$$\dot{\psi} = -\frac{\partial}{\partial p_i}(\psi \dot{p}_i) + D_{\mathrm{r}} \frac{\partial^2 \psi}{\partial p_i^2}; \tag{2.8}$$

here a superposed dot represents the material derivative and D_{r} is the rotary diffusivity due to Brownian motion.

Let $\boldsymbol{D}$ and $\boldsymbol{W}$ denote the deformation rate and spin tensors; these are defined by

$$\boldsymbol{D} = \frac{1}{2}(\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^{\mathrm{T}}), \tag{2.9}$$

$$\boldsymbol{W} = \frac{1}{2}(\nabla \boldsymbol{v} - (\nabla \boldsymbol{v})^{\mathrm{T}}). \tag{2.10}$$

Next, use is made of an equation for the motion of a single rigid ellipsoidal particle in a fluid; the orientation of the particle is determined from

$$\dot{\boldsymbol{p}} = \boldsymbol{W}\boldsymbol{p} + \lambda[\boldsymbol{D}\boldsymbol{p} - (\boldsymbol{p} \cdot \boldsymbol{D}\boldsymbol{p})\boldsymbol{p}] - \frac{D_{\mathrm{r}}}{\psi}\frac{\partial \psi}{\partial \boldsymbol{p}} \tag{2.11}$$

in which

$$\lambda = \frac{r^2 - 1}{r^2 + 1}, \tag{2.12}$$

where r is the fiber aspect ratio (Eq. (2.1)). The case $D_r = 0$ corresponds to that in which rotary diffusion is assumed absent, and the equation reduces to that obtained by Jeffery [19] for an ellipsoid of revolution in a dilute suspension.

For large particles, Folgar and Tucker [20] have proposed the empirical relation

$$D_r = 2\sqrt{2}\, C_I |\boldsymbol{D}|,$$

where C_I is a constant known as the interaction coefficient. The magnitude $|\boldsymbol{D}|$ of a tensor $\boldsymbol{D}$ is defined by $|\boldsymbol{D}| = (\boldsymbol{D}{:}\boldsymbol{D})^{1/2}$, and the inner product $\boldsymbol{A}{:}\boldsymbol{B}$ of two second-order tensors by $\boldsymbol{A}{:}\boldsymbol{B} = A_{ij}B_{ij}$. Since the magnitude of the interaction coefficient plays no role in the developments that follow, without loss of generality we henceforth set

$$D_r = |\boldsymbol{D}|. \tag{2.13}$$

The use of Eqs. (2.3), (2.8), (2.11) and (2.13) allows ψ to be eliminated, and we obtain as a result the evolution equation

$$\dot{\boldsymbol{A}} + \boldsymbol{AW} - \boldsymbol{WA} - \lambda(\boldsymbol{DA} + \boldsymbol{AD} - 2\mathscr{A}\boldsymbol{D}) - |\boldsymbol{D}|\,(\boldsymbol{I} - n\boldsymbol{A}) = \boldsymbol{0} \tag{2.14}$$

for the orientation tensor, for a problem posed in $\mathbb{R}^n$. As before a superposed dot denotes the material derivative, so that

$$\dot{\boldsymbol{A}} = \frac{\partial \boldsymbol{A}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{A}. \tag{2.15}$$

2.4. Closure approximations

Unfortunately Eq. (2.14) contains the tensor $\mathscr{A}$; in fact it is a feature of such evolution equations that the equation for an orientation tensor of a particular rank contains the tensor of the next (even) rank up. In order to circumvent this problem a closure approximation is used, in terms of which $\mathscr{A}$ is approximated by a function of the second-order tensors $\boldsymbol{A}$. Various such approximations are possible, and are the subject of the work by Advani and Tucker [18]. The simplest are the linear and quadratic approximations $\mathscr{A}^{\mathrm{L}}$ and $\mathscr{A}^{\mathrm{Q}}$; the linear approximation is most conveniently defined by its action on an arbitrary symmetric second-order tensor $\boldsymbol{D}$ and is given by

$$\mathscr{A}^{\mathrm{L}}\boldsymbol{D} = -\frac{1}{35}[(\mathrm{tr}\,\boldsymbol{D})\mathrm{I} + 2\boldsymbol{D}] + \frac{1}{7}[(\mathrm{tr}\,\boldsymbol{D})\boldsymbol{A} + 2\boldsymbol{AD} + 2\boldsymbol{DA} + (\boldsymbol{A}{:}\boldsymbol{D})\boldsymbol{I}]. \tag{2.16}$$

Since we shall be dealing with incompressible fluids ($\mathrm{tr}\,\boldsymbol{D} = 0$), this relation specializes to

$$\mathscr{A}^{\mathrm{L}}\boldsymbol{D} = -\frac{2}{35}\boldsymbol{D} + \frac{1}{7}[2\boldsymbol{AD} + 2\boldsymbol{DA} + (\boldsymbol{A}{:}\boldsymbol{D})\boldsymbol{I}]. \tag{2.17}$$

Note that $\mathscr{A}^{\mathrm{L}}$ satisfies the symmetry properties (Eq. (2.6)). The linear approximation is exact for random distributions of fibers, for which

$$A = \begin{bmatrix} 1/n & 0 & \dots & 0 \\ 0 & 1/n & \dots & 0 \\ \dots & & & \\ 0 & \dots & 0 & 1/n \end{bmatrix}$$

for a problem in $\mathbb{R}^n$.

The quadratic approximation is defined simply by

$$\mathscr{A}^{\mathrm{Q}} = \boldsymbol{A} \otimes \boldsymbol{A}. \tag{2.18}$$

This approximation is exact for fully aligned fibers; for example, in the case of fibres aligned parallel to the x_1-axis,

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

However, in general the quadratic approximation does not generally possess all the symmetries (Eq. (2.6)) present in the definition of $\mathscr{A}$.

Each approximation is suitable for only a range of physical situations, and there are of course situations in which neither is a good approximation. One attempt at remedying this situation has been to construct a weighted average of the linear and quadratic closure approximations; the resulting approximation, known as the hybrid approximation $\mathscr{A}^{\mathrm{H}}$, leads to stable dynamics for a wide range of flow fields and orientations, and is defined by

$$\mathscr{A}^{\mathrm{H}} = (1-f)\mathscr{A}^{\mathrm{L}} + f\mathscr{A}^{\mathrm{Q}}, \text{ for } 0 \le f \le 1. \tag{2.19}$$

The quantity f is a function of $\boldsymbol{A}$, so that the hybrid approximation is not a linear combination of the linear and quadratic approximations. One measure used in practice is [18]

$$f(\boldsymbol{A}) = 1 - N \det \boldsymbol{A}, \tag{2.20}$$

in which N equals 4 for planar flows and 27 for fully three-dimensional situations.

Yet other closure approximations have been proposed, and many of these are reviewed in [18]. More recently, Cintra and Tucker [21] have introduced the notion of an orthotropic closure approximation. This closure exploits the orthotropy of $\mathscr{A}$ to arrive at approximations that are expressed in terms of the eigenvalues of $\boldsymbol{A}$. Numerical simulations using the resulting closures yield results that are encouragingly accurate, in a wide range of flows.

2.5. Constitutive equation for the stress [4]

Coupling of the fluid and fiber motions leads to a modification of the usual constitutive equation for incompressible Newtonian fluids, in which the stress tensor $\boldsymbol{T}$ is given by

$$\boldsymbol{T} = -p\boldsymbol{I} + 2\mu\boldsymbol{D} + \boldsymbol{T}^{\mathrm{E}}, \tag{2.21}$$

where p is the pressure and μ the solvent viscosity. The extra stress $\boldsymbol{T}^{\mathrm{E}}$ is found by solving for the stress field around a single particle, which is assumed to be massless. The traction of the particle surface is then used to evaluate the particle contribution to the stress, and all particle orientations are then averaged to obtain the stress on the continuum scale. There are various theories which carry out this process, and all lead to expressions of the form

$$\boldsymbol{T}^E = 2\mu h[E\mathscr{A}\boldsymbol{D} + B(\boldsymbol{DA} + \boldsymbol{AD}) + H\boldsymbol{D} + 2FD_{\mathrm{r}}\boldsymbol{A}] \tag{2.22}$$

or, in component form,

$$T^{\mathrm{E}}_{ij} = 2\mu h[E\mathscr{A}_{ijkl}D_{kl} + B(D_{ik}A_{kj} + A_{ik}D_{kj}) + HD_{ij} + 2FD_{\mathrm{r}}A_{ij}] \tag{2.23}$$

in which h is the particle volume fraction and E, B, H and F are positive material constants. The term involving D_{r} accounts for Brownian motion, and is significant when particles are near-molecular size but negligible for the orders of magnitude encountered in reinforced polymers. This term is therefore neglected henceforth, and the stress is expressed in the form

$$\boldsymbol{T} = -p\boldsymbol{I} + 2\mu_I\boldsymbol{D} + \boldsymbol{S}, \tag{2.24}$$

in which

$$\boldsymbol{S} = 2\mu_I[N_{\mathrm{p}}\mathscr{A}\boldsymbol{D} + N_{\mathrm{S}}(\boldsymbol{DA} + \boldsymbol{AD})], \tag{2.25}$$

and

$$\mu_I = \mu(1 + hH), \qquad N_{\mathrm{p}} = \frac{hE}{1 + hH}, \qquad N_{\mathrm{s}} = \frac{hB}{1 + hH}. \tag{2.26}$$

The constants N_{p} and N_{s} are known as the particle number and shear number, and are both positive.

The fluid containing the fiber suspensions is assumed to occupy a bounded domain $\Omega \subset R^n$, where $n = 2$ or 3, with boundary Γ, and (as already observed) is assumed to be incompressible. The constitutive equations are supplemented by the equation of conservation of linear momentum

$$\rho\left(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right) - \mathrm{div}\,\boldsymbol{T} = \rho\boldsymbol{b}, \tag{2.27}$$

in which ρ is the mass density and $\boldsymbol{b}$ the body force per unit mass, and conservation of mass (the continuity equation)

$$\mathrm{div}\,\boldsymbol{v} = 0. \tag{2.28}$$

In addition we have to specify boundary conditions; we assume for convenience the non-slip condition

$$\boldsymbol{v} = 0 \ \text{ on } \ \Gamma \tag{2.29}$$

(no boundary condition is specified in respect of the orientation tensor) and the initial conditions

$$\boldsymbol{v}(\boldsymbol{x}, 0) = \boldsymbol{v}_0, \qquad \boldsymbol{A}(\boldsymbol{x}, 0) = \boldsymbol{A}_0 \ \text{ on } \ \bar{\Omega}. \tag{2.30}$$

3. The linear approximation and instability of the rest state

In the previous section we have introduced different types of approximations for the fourth-order tensor $\mathscr{A}$, as well as the material constants μ_I, N_p and N_s. As should be expected, the well-posedness of the initial-boundary value problem formulated in Section 2 may depend on the type of approximation used, and on whether or not these constants obey suitable restrictions (see [22] for analogous situations for fluids of second grade). The objective of this section is to show that for the linear closure approximation $\mathscr{A} = \mathscr{A}^\mathrm{L}$, some anomalous features occur if

$$\nu \equiv \mu_I\left(1 - \frac{2N_\mathrm{p}}{35}\right) < 0. \tag{3.1}$$

Specifically, if the condition (Eq. (3.1)) holds, it is shown that the rest state is nonlinearly unstable (in the sense of Liapounov), with respect to suitable norms. That is, there exists a perturbation which does not stay close to the rest state, no matter how small the initial data of the perturbation is.

To this end, we begin by defining the constant N by

$$N = 2N_\mathrm{p} + 7N_\mathrm{s},$$

and the tensor $\mathscr{S}$ by

$$\mathscr{S} = \frac{2\mu_I}{7}[N(\boldsymbol{AD} + \boldsymbol{DA}) + N_\mathrm{p}(\boldsymbol{A}{:}\boldsymbol{D})\boldsymbol{I}]; \tag{3.2}$$

then for the linear closure approximation, (Eq. (2.23)) with $\boldsymbol{b} \equiv \boldsymbol{0}$ takes the form

$$\rho\left(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right) - 2\nu \operatorname{div} \boldsymbol{D} + \nabla p = \operatorname{div} \mathscr{S}. \tag{3.3}$$

The pair $(\boldsymbol{v}, \boldsymbol{A})$ defined by

$$\boldsymbol{v} \equiv \boldsymbol{0}, \quad \boldsymbol{A} = \overline{\overline{\boldsymbol{A}}}$$

with $\overline{\overline{\boldsymbol{A}}}$ an arbitrary symmetric, constant tensor of trace 1, is a solution to (3.3), which we call the rest state.

Denote by $(\boldsymbol{v},\ \boldsymbol{A} \equiv \boldsymbol{a} + \overline{\overline{\boldsymbol{A}}})$ another possible solution to Eqs. (3.3), (2.14), (2.24) and (2.25). We shall say that the rest state is stable if and only if, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\sup_{x\in\Omega} |\boldsymbol{A}_0(x)| + \int_\Omega |\boldsymbol{v}_0(x)|^2 \mathrm{d}x < \delta \Rightarrow \sup_{x\in\Omega} |\boldsymbol{a}(x, t)| + \int_\Omega |\boldsymbol{v}(x, t)|^2 \mathrm{d}x < \varepsilon, \quad \forall t > 0. \tag{3.4}$$

Let

$$\overline{\overline{\boldsymbol{A}}} = \boldsymbol{e}_1 \otimes \boldsymbol{e}_1. \tag{3.5}$$

We have

Theorem 1. *Assume that Eq. (3.1) holds. Then the rest state $\overline{\overline{\boldsymbol{A}}}$ given in Eq. (3.5) is unstable.*

Proof. As a perturbation to the velocity field, we choose the planar field

$$\boldsymbol{v}(\boldsymbol{x},t) = v_2(x_2,x_3,t)\boldsymbol{e}_2 + v_3(x_2,x_3,t)\boldsymbol{e}_3, \tag{3.6}$$

together with the corresponding pressure field p and orientation $\boldsymbol{a}$. We note in particular from Eqs. (3.5) and (3.6) that

$$\overline{\overline{\boldsymbol{A}}}\boldsymbol{D}(\boldsymbol{v}) = 0.$$

Taking the scalar product of both sides of Eq. (3.3) with $\boldsymbol{v}$, integrating by parts over Ω and using Eqs. (2.28) and (2.29), we obtain, with $\boldsymbol{D} \equiv \boldsymbol{D}(\boldsymbol{v})$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\boldsymbol{v}(t)|^2\,\mathrm{d}x = \bar{\nu}\int_\Omega |\boldsymbol{D}(t)|^2\,\mathrm{d}x - \frac{1}{\rho}\int_\Omega \mathscr{S} : \boldsymbol{D}\,\mathrm{d}x, \tag{3.7}$$

where $\bar{\nu} = 2|\nu|/\rho$. Using Eq. (3.2) and setting

$$a = \sup_{x\in\Omega} |\boldsymbol{a}(x,\,t)|$$

we at once find that

$$\left(\int_\Omega |\mathscr{S}|^2\,\mathrm{d}x\right)^{1/2} \le \frac{2\mu_I}{7}\mathscr{C}a\left(\int_\Omega |\boldsymbol{D}|^2\,\mathrm{d}x\right)^{1/2}, \tag{3.8}$$

where $\mathscr{C}^2 = 3N_\mathrm{p}^2 + 4N_\mathrm{p}N + 4N^2$. Employing the Cauchy-Schwarz inequality on the right-hand side of Eq. (3.7) and then using Eq. (3.8) we thus have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\boldsymbol{v}(t)|^2\,\mathrm{d}x \ge \left[\bar{\nu} - \frac{2\mu_I}{7\rho}\mathscr{C}a\right]\int_\Omega |\boldsymbol{D}(t)|^2\,\mathrm{d}x. \tag{3.9}$$

Assume now that the rest state is stable. Subsequently, in view of Eq. (3.4), we may choose ε sufficiently small so that the condition

$$\bar{\nu} - \frac{2\mu_I}{7\rho}\mathscr{C}a \ge \frac{1}{2}\bar{\nu} > 0$$

holds. Thus, Eq. (3.9) furnishes

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\boldsymbol{v}(t)|^2\,\mathrm{d}x \ge \bar{\nu}\int_\Omega |\boldsymbol{D}(t)|^2\,\mathrm{d}x. \tag{3.10}$$

Since div $\boldsymbol{v} = 0$ we have

$$\int_\Omega |\boldsymbol{D}|^2\,\mathrm{d}x = \frac{1}{2}\int_\Omega |\nabla\boldsymbol{v}|^2\,\mathrm{d}x$$

and so, using the Poincaré–Friedrichs inequality

$$\int_\Omega |\nabla\boldsymbol{v}|^2\,\mathrm{d}x \ge c_0\int_\Omega |\boldsymbol{v}|^2\,\mathrm{d}x, \tag{3.11}$$

we obtain

$$\int_\Omega |\boldsymbol{D}|^2\,\mathrm{d}x \ge \frac{1}{2}c_0\int_\Omega |\boldsymbol{v}|^2\,\mathrm{d}x.$$

From this inequality, Eq. (3.10) and Gronwall's lemma, we deduce that

$$\int_\Omega |\boldsymbol{v}(t)|^2 \,\mathrm{d}x \geq \mathrm{e}^{\bar{\nu} c_0/2\, t} \int_\Omega |\boldsymbol{v}_0|^2 \,\mathrm{d}x,$$

which contradicts the assumption of stability. This proves the theorem. □

Given that $\mu_\mathrm{I} > 0$ at all times, this theorem suggests that for the case of the linear approximation $\mathscr{A} = \mathscr{A}^\mathrm{L}$, the model we are using can furnish physically reasonable predictions only when

$$N_\mathrm{p} \leq 35/2.$$

We emphasize that these anomalous features do not appear when the approximations Eq. (2.18) or Eq. (2.19) are used, due to the fact that in these cases the tensor field $\mathscr{A}\boldsymbol{D}$ becomes purely nonlinear in the pair $(\boldsymbol{A}, \boldsymbol{D})$, in the sense that its Fréchet derivative vanishes at (0, 0).

Bearing in mind the unstable nature of the linear closure approximation, and noting also the fact that use of the quadratic closure leads to results that are generally plausible, we will henceforth take as the model problem that which is based on the quadratic closure approximation. It is worth noting, though, that by using the same techniques it should be possible to derive analogous results for the hybrid approximation ($\mathscr{A} = \mathscr{A}^\mathrm{H}$), and even for the linear approximation ($\mathscr{A} = \mathscr{A}^\mathrm{L}$), provided the material constants satisfy the condition

$$N_\mathrm{p} \leq 35/2.$$

Remarks 1. The particle number N_p is a function of particle aspect ratio and volume fraction; for example, for slender particles an asymptotic approximation is [9]

$$N_\mathrm{p} = \frac{hr^2}{2(\ln 2r - 1.5)}.$$

From this relationship, it may be shown that the constraint $N_\mathrm{p} \leq 35/2$ corresponds to the requirement that the suspension be at most semi-dilute (cf. Eq. (2.1) and Fig. 1). Such a

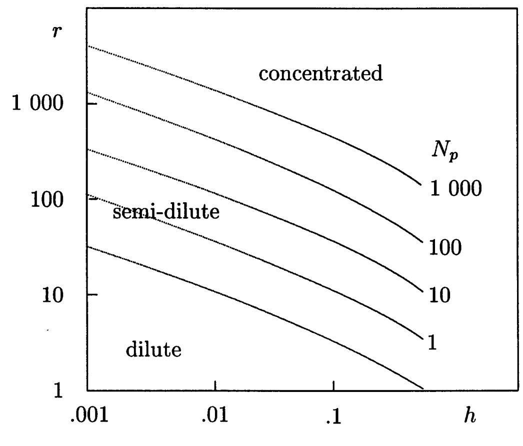

Fig. 1. Particle number as a function of volume fraction and aspect ratio.

limitation is in fact consistent with the theory as a whole, since the governing equations are based on the assumption of a low incidence of inter-particle contact (that is, the dilute and semi-dilute ranges).

Remark 2. The linear closure approximation exhibits anomalous behavior also in another context. Numerical studies of Eq. (2.14) have been carried out in [17] for simple steady shear flows, and for the linear, quadratic and hybrid closure approximations. These studies indicate that, for small values of the interaction constant C_{I} at least, and for the quadratic and hybrid approximations, the solutions $\boldsymbol{A}(t)$ settle down to a steady state, with varying degrees of accuracy when compared with the exact solution. For the linear approximation, however, the solution is unrealistic in that it exceeds unity, and furthermore it is oscillatory in nature. Investigations by Reddy and Mitchell [11] provide further numerical evidence of this phenomenon. This oscillatory behavior, and its possible relationship to the instability which is the subject of Theorem 1, have yet to be investigated theoretically.

4. Formulation of the problem

The complete problem may now be summarized as follows: given the initial data, the density ρ and the body force $\boldsymbol{b}$, find $\boldsymbol{v}$, $\boldsymbol{A}$ and p which satisfy Eqs. (2.14), (2.18), (2.24), (2.25), (2.26), (2.27), (2.28), (2.29) and (2.30). Before embarking on an analysis of this problem, it is useful to carry out the following additive decomposition of $\boldsymbol{A}$: we set

$$\boldsymbol{A} = \tilde{\boldsymbol{A}} + \boldsymbol{A}^*, \tag{4.1}$$

in which $\tilde{\boldsymbol{A}} = \boldsymbol{A} - \boldsymbol{A}^*$ and $\boldsymbol{A}^*$ is any constant symmetric, positive semi-definite tensor having unit trace: that is,

$$A^*_{ji} = A^*_{ij}, \;\; tr\, A^* = 1, \;\; \boldsymbol{z} \cdot \boldsymbol{A}^*\boldsymbol{z} \geq 0 \quad \forall \text{ vectors } \boldsymbol{z}. \tag{4.2}$$

The property of positive semi-definiteness implies of course that $\boldsymbol{A}^*$ has non-negative eigenvalues, while its unit trace property implies that $\tilde{\boldsymbol{A}}$ is traceless:

$$\mathrm{tr}\, \tilde{\boldsymbol{A}} = 0. \tag{4.3}$$

For convenience, and without any loss in generality, we henceforth choose

$$\boldsymbol{A}^* = \tfrac{1}{n}\boldsymbol{I} \tag{4.4}$$

for a problem posed in $\mathbb{R}^n$.

We will confine attention to the quadratic closure approximation, for which case we have, bearing in mind Eq. (4.4) and the incompressibility condition,

$$\mathscr{A}\boldsymbol{D} \equiv \mathscr{A}^Q\boldsymbol{D} = (\tilde{\boldsymbol{A}} + \boldsymbol{A}^*) \otimes (\tilde{\boldsymbol{A}} + \boldsymbol{A}^*)\boldsymbol{D} = (\tilde{\boldsymbol{A}}{:}\boldsymbol{D})(\tilde{\boldsymbol{A}} + \tfrac{1}{n}\boldsymbol{I}). \tag{4.5}$$

The problem is now nondimensionalized: with dimensionless quantities denoted by a superposed bar, we set

$$\bar{\boldsymbol{x}} = \frac{x}{L}, \quad \bar{\boldsymbol{v}} = \frac{\boldsymbol{v}}{V}, \quad \bar{\boldsymbol{t}} = \frac{tV}{L},$$
$$\bar{p} = \frac{pL}{\mu V}, \quad \bar{\boldsymbol{b}} = \frac{\boldsymbol{b}L^2}{\mu V}, \quad \bar{\boldsymbol{S}} = \frac{\boldsymbol{S}L}{\mu V},$$

where L and V are, respectively, a characteristic length and velocity. We also introduce the Reynolds number $\mathrm{Re} = \rho VL/\mu$.

We now substitute the decomposition Eq. (4.1) in Eqs. (2.14), (2.19) and (2.25); then, removing the bars from dimensionless quantities and writing $\boldsymbol{A}$ for $\tilde{\boldsymbol{A}}$, without any danger of ambiguity, we obtain, using also Eq. (4.5), the system of dimensionless equations

$$\left.\begin{array}{c} \mathrm{Re}(\boldsymbol{v}' + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}) + \nabla p - \Delta \boldsymbol{V} = \boldsymbol{b} + \mathrm{div}\,\boldsymbol{S} \\ \mathrm{div}\,\boldsymbol{v} = 0 \\ \gamma\left[N_{\mathrm{p}}(\boldsymbol{A}:\boldsymbol{D})(\boldsymbol{A} + \frac{1}{n}\boldsymbol{I}) + N_{\mathrm{s}}\left(\boldsymbol{D}\boldsymbol{A} + \boldsymbol{A}\boldsymbol{D} + \frac{2}{n}\boldsymbol{D}\right)\right] = \boldsymbol{S} \\ \boldsymbol{A}' + (v\cdot\nabla)\boldsymbol{A} + (\boldsymbol{A}\boldsymbol{W} - \boldsymbol{W}\boldsymbol{A}) - \lambda\left(\boldsymbol{D}\boldsymbol{A} + \boldsymbol{A}\boldsymbol{D} + \frac{2}{n}\boldsymbol{D} - 2(\boldsymbol{A}:\boldsymbol{D})(\boldsymbol{A} + \frac{1}{n}\boldsymbol{I})\right) + n|\boldsymbol{D}|\boldsymbol{A} = \boldsymbol{0} \end{array}\right\} \text{ in } \Omega_T, \tag{4.6}$$

together with the boundary and initial conditions Eq. (2.29) and Eq. (2.30). Here $\gamma = 2\mu_I/\mu$, and $(\cdot)' \equiv \partial(\cdot)/\partial t$.

4.1. Function spaces

We make use of the Lebesgue spaces $L^p(\Omega)$, $1 \le p \le \infty$, with norms defined in the usual way and denoted by $\|\cdot\|_{L^p}$. We will also require the Sobolev spaces $H^k(\Omega)$ for integers $k = 0, 1, \ldots$ These are Hilbert spaces endowed with the Sobolev inner product $(\cdot,\cdot)_{H^k}$ defined by

$$(u, v)_{H^k} = \sum_{|\alpha| \le k} \int_\Omega D^\alpha u D^\alpha v \,\mathrm{d}x,$$

in which $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index with each α_i a nonnegative integer; $|\alpha| = \alpha_1 + \ldots + \alpha_n$ and $D^\alpha u$ denotes the partial derivative $\partial^{|\alpha|}u/\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}$.

The Sobolev norm is defined by $\|u\|_{H^k} = (u, u)_{H^k}^{1/2}$, while the H^k-seminorm is defined by $|u|_{H^k}^2 = \Sigma_{|\alpha|=k}\|D^\alpha u\|_{L^2}^2$. The space $H_0^1(\Omega)$ of functions in $H^1(\Omega)$ which vanish on Γ in the sense of traces will also be required, as will its topological dual space $H^{-1}(\Omega)$; duality pairing of two elements $\ell \in H^{-1}(\Omega)$ and $u \in H_0^1(\Omega)$ will be denoted by $\langle \ell, u\rangle$. By the Poincaré–Friedrichs inequality Eq. (3.11), the H^1 seminorm $|u|_{H^1} = \|\nabla u\|_{L^2}$ is a norm on $H_0^1(\Omega)$, equivalent to the natural H^1-norm.

Spaces of vector- or tensor-valued functions which have components in one of the spaces introduced above, will be denoted by the same symbol in boldface; for example, $\boldsymbol{H}^1(\Omega)$ will denote the space of vector- or tensor-valued functions with components in $H^1(\Omega)$. The norms on these spaces are defined as the sums of relevant norms of components, and will be denoted in the same way as for scalar-valued functions, without any danger of ambiguity. We will require the special spaces

$$H=\{\boldsymbol{v}: v_i\in L^2(\Omega),\ \mathrm{div}\ \boldsymbol{v}=0\ \text{ in }\ \Omega,\ \boldsymbol{vn}=0\ \text{ on }\Gamma\}$$

$$V=\{\boldsymbol{v}: v_i\in H_0^1(\Omega),\ \mathrm{div}\ \boldsymbol{v}=0\ \text{ in }\ \Omega\},$$

$$X=\{\boldsymbol{A}: A_{ij}\in L^2(\Omega),\ A_{ji}=A_{ij},\ A_{ii}=0\ \text{ a.e. in }\ \Omega\},$$

equipped, respectively, with the norms $\|\cdot\|_H\equiv\|\cdot\|_{L^2}$, $\|\cdot\|_V\equiv|\cdot|_{H^1}$ and $\|\cdot\|_Y\equiv\|\cdot\|_{L^2}$. We also set

$$IH^m=H^m(\Omega)\bigcap X.$$

The orthogonal projection of $\boldsymbol{L}^2(\Omega)$ onto H is denoted by P, and we define the extended Stokes operator $\mathscr{L}$ by

$$\mathscr{L}\boldsymbol{v}=-\left(1+\frac{2}{n}\gamma N_s\right)P\Delta\boldsymbol{v}; \tag{4.7}$$

$\mathscr{L}$ has domain $\mathrm{D}(\mathscr{L})=V\bigcap\boldsymbol{H}^2(\Omega)$, and is equipped with the norm $\|\boldsymbol{v}\|_{D(\mathscr{L})}=\|\mathscr{L}\boldsymbol{v}\|_{L^2}$, which [23] is equivalent to the natural H^2-norm.

With the aid of the projection P we introduce bilinear mappings $b(\cdot,\cdot)$ and $B(\cdot,\cdot)$ corresponding to the convective terms in Eq. $(4.6)_1$ and Eq. $(4.6)_4$, according to

$$b(\boldsymbol{v},\boldsymbol{w})=P(\boldsymbol{v}\cdot\nabla)\boldsymbol{w},$$

$$B(\boldsymbol{v},\boldsymbol{A})=\boldsymbol{v}\cdot\nabla\boldsymbol{A},$$

for sufficiently smooth vector-valued functions $\boldsymbol{v}$, $\boldsymbol{w}$, and tensor-valued functions $\boldsymbol{A}$, and the map $G(\cdot,\cdot,\cdot)$ by

$$G(\boldsymbol{A},\boldsymbol{W},\boldsymbol{D})=\boldsymbol{WA}-\boldsymbol{AW}-\lambda\left[2(\boldsymbol{A}:\boldsymbol{D})\left(\boldsymbol{A}+\frac{1}{n}\boldsymbol{I}\right)-\boldsymbol{DA}-\boldsymbol{AD}-\frac{2}{n}\boldsymbol{D}\right]-n|\boldsymbol{D}|\boldsymbol{A}.$$

We note that for any $\boldsymbol{v}\in V$ and for a tensor ϕ of arbitrary rank the identity

$$\int_\Omega[(\boldsymbol{v}\nabla)\boldsymbol{\phi}]:\boldsymbol{\phi}\mathrm{d}x\equiv\int_\Omega v_i\boldsymbol{\phi}_{,i}:\boldsymbol{\phi}\mathrm{d}x=\frac{1}{2}\int_\Omega v_i(\boldsymbol{\phi}:\boldsymbol{\phi})_{,i}=0 \tag{4.8}$$

holds. In particular, for the case in which $\boldsymbol{\phi}$ is the second-rank tensor $\boldsymbol{A}$, we have

$$(B(\boldsymbol{v},\boldsymbol{A}),\boldsymbol{A})_{L^2}=0. \tag{4.9}$$

We introduce some standard notation. Let X be a Banach space and T a positive number; then the space $C^m([0,T],X)$ $(m=0,1,\ldots)$ consists of all continuous functions u from $[0,T]$ to X that have continuous derivatives up to and including those of order m. This is a Banach space when endowed with the norm

$$\|u\|_{C^m([0,T],X)}=\sum_{k=0}^{m}\max_{0\le t\le T}|u^{(k)}(t)|$$

where $|u^{(k)}(t)|$ denotes the kth time derivative of u. We write $C([0,T],X)$ for the case $m=0$.

For $1 \leq p < \infty$ the space $L^p(0, T; X)$ consists of all measurable functions u from $[0, T]$ to X for which

$$\|u\|_{L^p(0,\,T,\,X)} \equiv \left(\int_0^T \|u(t)\|_X^p \, \mathrm{d}t \right)^{1/p} < \infty.$$

This is a Banach space with the norm $\|u\|_{L^p(0,\,T,\,X)}$. The space $L^\infty(0, T; X)$ consists of all measurable functions u from $[0, T]$ to X which are essentially bounded. This is a Banach space with the norm

$$\|u\|_{L^\infty(0,\,T,\,X)} \equiv \operatorname{ess\,sup} \|u(t)\|_X.$$

The problem to be considered now takes the following form:

Problem $\mathscr{P}$. Given suitable functions $\boldsymbol{v}_0$ and $\boldsymbol{A}_0$, find

$$\boldsymbol{v}(\cdot\,, t) \in V \text{ and } \boldsymbol{A}(\cdot\,, t) \in I\!H^2$$

such that for a.a. $t \in (0, T)$,

$$\mathrm{Re}[\boldsymbol{v}' + b(\boldsymbol{v}, \boldsymbol{v})] + \mathscr{L}\boldsymbol{v} = P(\boldsymbol{b} + \operatorname{div} \tilde{\boldsymbol{S}}), \tag{4.10}$$

$$\gamma \left[N_\mathrm{p}(\boldsymbol{A}:\boldsymbol{D})(\boldsymbol{A} + \frac{1}{n}\boldsymbol{I}) + N_\mathrm{s}(\boldsymbol{D}\boldsymbol{A} + \boldsymbol{A}\boldsymbol{D}) \right] = \tilde{\boldsymbol{S}}, \tag{4.11}$$

$$\boldsymbol{A}' + B(\boldsymbol{v}, \boldsymbol{A}) - G(\boldsymbol{A}, \boldsymbol{W}, \boldsymbol{D}) = 0, \tag{4.12}$$

$$\boldsymbol{v}(0) = \boldsymbol{v}_0, \;\; \boldsymbol{A}(0) = \boldsymbol{A}_0. \tag{4.13}$$

The aim of this paper is to show that Problem $\mathscr{P}$ admits a unique solution, provided that T and the data are sufficiently small.

5. Existence and uniqueness of solutions to two auxiliary problems

The problem of showing that Problem $\mathscr{P}$ has a unique regular solution (local in time and for small data) is approached using a fixed point argument. The procedure followed is inspired by that used in [12], in the context of the problem of certain viscoelastic flows.

The strategy to be adopted is as follows:

Step 1. We consider the Stokes problem

$$\text{Problem I.} \begin{cases} \boldsymbol{v}(\cdot\,, t) \in V, & \text{a.e. in } (0, T), \\ \mathrm{Re}\, \boldsymbol{v}' + \mathscr{L}\boldsymbol{v} = \boldsymbol{F} & \text{a.e. in } (0, T), \\ \boldsymbol{v}(0) = \boldsymbol{v}_0, \end{cases} \tag{5.1}$$

in which $\boldsymbol{F}$ is a given external force. Results on the existence and uniqueness of a solution to this linear problem, as well as estimates for the solution $\boldsymbol{v}$ in terms of $\boldsymbol{F}$ and the initial data, have been previously established, so that it will be necessary merely to recall these.

Step 2. The next step entails the solution of a transport problem

$$\text{Problem II.}\begin{cases} \boldsymbol{A}(\cdot,t)\in I\!H^2, & \text{a.e. in } (0,T),\\ \boldsymbol{A}'+B(\bar{\boldsymbol{v}},\boldsymbol{A})-\bar{G}(\boldsymbol{A},\bar{\boldsymbol{A}})=\boldsymbol{0}, & \text{a.e. in } (0,T),\\ \boldsymbol{A}(0)=\boldsymbol{A}_0, \end{cases} \tag{5.2}$$

based on Eq. (4.12), in which $\bar{\boldsymbol{v}}$ is a specified velocity field, and $\bar{G}$ is a suitable linearization of the function G, defined by

$$\bar{G}(\boldsymbol{A},\bar{\boldsymbol{A}})=\bar{\boldsymbol{W}}\boldsymbol{A}-\boldsymbol{A}\bar{\boldsymbol{W}}-\lambda\left[2(\boldsymbol{A}:\boldsymbol{D})(\bar{\boldsymbol{A}}+\frac{1}{n}\boldsymbol{I})-\bar{\boldsymbol{D}}\boldsymbol{A}-\boldsymbol{A}\bar{\boldsymbol{D}}-\frac{2}{n}\bar{\boldsymbol{D}}\right]-n|\bar{\boldsymbol{D}}|\,\boldsymbol{A}. \tag{5.3}$$

The quantity $\bar{\boldsymbol{A}}$ is a specified tensor-valued field from the class X, and $\bar{\boldsymbol{W}}$ and $\bar{\boldsymbol{D}}$ are the spin and deformation rate tensors corresponding to $\bar{\boldsymbol{v}}$.

Step 3. Consider the map Φ: $(\bar{\boldsymbol{v}},\bar{\boldsymbol{A}})\rightarrow(\boldsymbol{v},\boldsymbol{A})$, in which $\boldsymbol{v}$ and $\boldsymbol{A}$ are the solutions to Problems I and II, respectively, and in which $\boldsymbol{F}$ in Problem I is defined by

$$\boldsymbol{F}=-\mathrm{Re}\,b(\bar{\boldsymbol{v}},\bar{\boldsymbol{v}})+P\boldsymbol{b}+P\,\mathrm{div}\,\tilde{\boldsymbol{S}}(\bar{\boldsymbol{v}},\bar{\boldsymbol{A}}), \tag{5.4}$$

with $\tilde{\boldsymbol{S}}$ given by Eq. (4.11). A fixed point of Φ, in a suitable function class, is clearly a solution to Problem $\mathscr{P}$.

5.1. Solution of the Stokes problem

The solution to Problem I, as well as the relevant a priori estimate, may be immediately deduced from the solution to the standard Stokes problem which has been given in [12]; that result is reproduced here.

Lemma 2. *Assume that Γ is of class C^3, $\boldsymbol{F}\in L^2(0,T;\boldsymbol{H}^1)$, $\boldsymbol{F}'\in L^2(0,T;\boldsymbol{H}^{-1})$, and $\boldsymbol{v}_0\in D(\mathscr{L})$. Then the Stokes problem (Problem I) has a unique solution (v,p). Furthermore,*

$$\boldsymbol{v}\in L^2(0,T;\boldsymbol{H}^3),\ \boldsymbol{v}'\in L^2(0,T;V),\ p\in L^2(0,T;\boldsymbol{H}^2)$$

and there exists a constant C depending on Re and Ω, such that

$$\|\boldsymbol{v}\|_{L^2(0,T;H^3)\cap L^\infty(0,T;D(\mathscr{L}))}+\|\boldsymbol{v}'\|_{L^2(0,T;V)\cap L^\infty(0,T;H)}+\|p\|_{L^2(0,T;H^2)}$$
$$\le C[\|\mathscr{L}\boldsymbol{v}_0\|^2_{L^2}+\|\boldsymbol{F}\|^2_{L^2(0,T;H^1)}+\|\boldsymbol{F}'\|^2_{L^2(0,T;H^{-1})}+\|\boldsymbol{F}(0)\|^2_{L^2}]. \tag{5.5}$$

5.2. Solution of the transport problem

We turn next to Problem II, summarized in Step 2 above. The following result is valid.

Lemma 3. *Assume that Γ is of class C^1, $\bar{\boldsymbol{v}}\in L^2(0,T;\boldsymbol{H}^3)\cap L^\infty(0,T;D(\mathscr{L}))$, $\bar{\boldsymbol{A}}\in L^\infty(0,T;I\!H^3)$ and $\boldsymbol{A}_0\in I\!H^2$.* Assume also that

$$\|\bar{\boldsymbol{v}}\|_{L^2(0,T;H^3)}+\|\bar{\boldsymbol{v}}\|_{L^\infty(0,T;D(\mathscr{L}))}\le B_1,\ \ \|\bar{\boldsymbol{A}}\|_{L^\infty(0,T;H^2)}\le B_2 \tag{5.6}$$

with

$$B_2 \geq 2(B_1 + \|A_0\|_{H^2}). \tag{5.7}$$

Then there exists a constant $h > 0$ depending on B_1, B_2, Ω and the material constants, such that the problem Eq. (5.2), with $\bar{G}$ given by Eq. (5.3), admits one and only one solution A in the time interval (0, T^*), where

$$T^* \leq \min\{h, T\}$$

and

$$A \in L^\infty(0,T^*;IH^2),\ A' \in L^\infty(0,T^*;IH^1).$$

Furthermore,

$$\begin{aligned} &\|A\|_{L^\infty(0,\ T^*;\ H^2)} \leq B_2, \\ &\|A'\|_{L^\infty(0,\ T^*;\ H^1)} \leq B_3, \end{aligned} \tag{5.8}$$

where $B_3 \equiv B_1\mathscr{R}(B_2)$ and $\mathscr{R}(B_2)$ is a polynomial in B_2.

Proof. We first establish some suitable a priori estimates which lead to Eq. (5.8). These estimates can then be coupled with the Galerkin method, as suggested in [22], to show the existence of a solution satisfying the required properties.

Taking the inner product of Eq. $(5.2)_2$ with A, integrating over Ω and using Eq. (4.9), we arrive at the identity

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|A\|_{L^2}^2 = \int_\Omega \bar{G}(A, \bar{A}) : A \,\mathrm{d}x.$$

Next we apply the operator $\partial/\partial x_l$ to Eq. $(5.2)_2$, take the L^2-inner product of this equation with $A_{,l}$ and simplify, using Eq. (4.8), to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|A|_{H^1}^2 = -\int_\Omega \bar{v}_{k,\,l}A_{,k} : A_{,l}\,\mathrm{d}x + \int_\Omega (\bar{G}(A, \bar{A})_{,l} : A_{,l})\,\mathrm{d}x,$$

in which a subscript following a comma denotes partial differentiation with respect to that component. Finally, we apply the operator $\partial^2/\partial x_l \partial x_m$ to Eq. $(5.2)_2$, take the L^2-inner product of this equation with $A_{,lm}$, and simplify in the same way as above, to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|A|_{H^2}^2 = -\int_\Omega [\bar{v}_{k,\,lm}A_{,k} : A_{,lm} + 2\bar{v}_{k,l}A_{,km} : A_{,lm}]\mathrm{d}x + \int_\Omega (\bar{G}(A, \bar{A})_{,lm} : A_{,lm})\mathrm{d}x.$$

Adding these three equations, and recalling Eqs. (4.3) and (5.3), we find that

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|A\|_{H^2}^2 = &-\int_\Omega \{\bar{v}_{k,\,l}[A_{,k} : A_{,l} + 2A_{,km} : A_{,lm}] + \bar{v}_{k,lm}A_{,k} : A_{,lm}\}\mathrm{d}x - 2\lambda((A{:}D)\bar{A},A)_{H^2} \\ &+ \lambda\left(\bar{D}A + A\bar{D} + \frac{2}{n}\bar{D},A\right)_{H^2} + ((\bar{W}A - A\bar{W}),A)_{H^2} - n(|\bar{D}|A,A)_{H^2}. \end{aligned} \tag{5.9}$$

We now estimate the right-hand side of Eq. (5.9). To this end, we recall the Sobolev inequality

$$\max_{x\in\Omega}|u(x)| + \|u\|_{L^4} \leq c\|u\|_{H^2} \text{ for all } u \in H^2(\Omega) \tag{5.10}$$

which implies that

$$\left|\int_\Omega \bar{v}_{k,l}\boldsymbol{A}_{,k} : \boldsymbol{A}_{,l}\mathrm{d}x\right| \le c\|\bar{\boldsymbol{v}}\|_{H^3}\|\boldsymbol{A}\|^2_{H^2},$$

$$\left|\int_\Omega \bar{v}_{k,l}\boldsymbol{A}_{,km} : \boldsymbol{A}_{,lm}\mathrm{d}x\right| \le c\|\bar{\boldsymbol{v}}\|_{H^3}\|\boldsymbol{A}\|^2_{H^2},$$

$$\left|\int_\Omega \bar{v}_{k,lm}\boldsymbol{A}_{,k} : \boldsymbol{A}_{,lm}\mathrm{d}x\right| \le c\|\bar{\boldsymbol{v}}\|_{H^3}\|\boldsymbol{A}\|^2_{H^2}. \tag{5.11}$$

Likewise, from Eq. (5.10) we find that

$$|((\boldsymbol{A}:\boldsymbol{D})\bar{\boldsymbol{A}},\boldsymbol{A})_{H^2}| \le c\|\bar{\boldsymbol{v}}\|_{H^3}\|\boldsymbol{A}\|^2_{H^2}\|\bar{\boldsymbol{A}}\|_{H^2}),$$

$$|((\bar{\boldsymbol{W}}\boldsymbol{A} - \boldsymbol{A}\bar{\boldsymbol{W}},\boldsymbol{A})_{H^2}| \le c\|\bar{\boldsymbol{v}}\|_{H^3}\|\boldsymbol{A}\|^2_{H^2},$$

$$\left|((\bar{\boldsymbol{D}}\boldsymbol{A} + \boldsymbol{A}\bar{\boldsymbol{D}} + \frac{2}{n}\bar{\boldsymbol{D}}),\boldsymbol{A})_{H^2}\right| \le c\|\bar{\boldsymbol{v}}\|_{H^2}\|\boldsymbol{A}\|_{H^2}(1 + \|\boldsymbol{A}\|_{H^2}),$$

$$|(|\bar{\boldsymbol{D}}|\boldsymbol{A},\boldsymbol{A})_{H^2}| \le c\|\bar{\boldsymbol{v}}\|_{H^3}\|\boldsymbol{A}\|^2_{H^2}. \tag{5.12}$$

The constant c entering the estimates Eqs. (5.11) and (5.12) depends only on Ω, and on the material constants. By making use of Eqs. (5.11) and (5.12) in Eq. (5.9) we thus conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{A}\|^2_{H^2} \le c[1 + \|\boldsymbol{A}\|_{H^2} + \|\bar{\boldsymbol{A}}\|_{H^2}]\|\boldsymbol{A}\|_{H^2}\|\bar{\boldsymbol{v}}\|_{H^3}. \tag{5.13}$$

Setting $y = \|\boldsymbol{A}\|^2_{H^2}$, and using the assumption Eq. $(5.6)_2$, we find that

$$y'(t) \le M(t)y(t) + N(t) \tag{5.14}$$

with

$$M(t) = cg(t) + \tfrac{1}{2}(1 + B_2),\ \ N(t) = \tfrac{1}{2}(1 + B_2)[g(t)]^2,$$

and

$$g(t) = \|\bar{\boldsymbol{v}}(t)\|_{H^3}.$$

Integration of Eq. (5.14) with initial data y_0, and the use of Eq. $(5.6)_1$, lead to the inequality

$$y(t) \le (y_0 + B_1^2)\exp[cB_1\sqrt{t} + \tfrac{1}{2}(1 + B_2)t]. \tag{5.15}$$

Now, by assumption $y_0 + B_1^2 \le B_2^2/4$, and so, from Eq. (5.15) it follows that

$$y(t) \le \frac{B_2^2}{4}\exp[cB_1\sqrt{t} + \tfrac{1}{2}(1 + B_2)t].$$

The estimate Eq. $(5.8)_1$ then follows by choosing h such that

$$\exp[cB_1\sqrt{h} + \tfrac{1}{2}(1 + B_2)h] < 4.$$

The estimate Eq. $(5.8)_2$ is obtained directly by taking the H^1-norm of both sides of Eq. $(5.2)_2$ and using Eq. (5.10) and Eq. $(5.8)_1$. By employing the Galerkin method as in [22], we arrive at the desired existence result.

To show uniqueness, we assume that $A^{(1)}$, $A^{(2)}$ are two solutions corresponding to the same data, and set

$$A = A^{(1)} - A^{(2)},\ G_1(A, \bar{A}) = \bar{G}(A_1, \bar{A}) - \bar{G}(A_2, \bar{A}).$$

From Eq. (5.2), we obtain

$$A' + B(\bar{v}, A) + G_1(A, \bar{A}) = 0, \tag{5.16}$$

and from Eqs. (5.6) and (5.11) we easily find that

$$|(G_1(A, \bar{A}), A)_{L^2}| \le c\|\bar{v}\|_{H^3}\|A\|^2_{L^2}$$

with c a positive constant depending only on B_2, Ω and the material constants. Thus, taking the L^2-inner product of both sides of Eq. (5.16) with A and using this latter estimate along with Eq. (4.9), we obtain

$$\frac{d}{dt}\|A\|^2_{L^2} - 2c\|\bar{v}\|_{H^3}\|A\|^2_{L^2} \le 0. \tag{5.17}$$

Since

$$\bar{v} \in L^2(0, T^*; H^3) \tag{5.18}$$

we may apply Gronwall's lemma to Eq. (5.17) to deduce that $A = 0$, a.e. in Ω^*_T. This establishes the uniqueness of the solution.

It remains to show that the solution A belongs to X; that is, we must show that

(a) $A^T = A$; and

(b) $\operatorname{tr} A = 0$.

Substituting for $\bar{G}$ in Eq. (5.2), using the definition Eq. (5.3) we obtain

$$A' + B(\bar{v}, A) - \bar{W}A + A\bar{W} + \lambda[2(A{:}\bar{D})(\bar{A} + \tfrac{2}{n}I) - \bar{D}A - A\bar{D} - \tfrac{2}{n}\bar{D}] + n|\bar{D}|A = 0. \tag{5.19}$$

Next, we take the transpose of this equation to find that

$$A^{T'} + B(\bar{v}, A^T) - A^T\bar{W} + \bar{W}A^T + \lambda[2(A{:}\bar{D})(\bar{A}^T + \tfrac{2}{n}I) - \bar{D}A^T - A^T\bar{D} - \tfrac{2}{n}\bar{D}] + n|\bar{D}|A^T = 0. \tag{5.20}$$

Defining $Q = A^T - A$, and subtracting Eq. (5.19) from Eq. (5.20), we obtain

$$Q' + B(\bar{v}, Q) + Q\bar{W} - \bar{W}Q$$
$$+ \lambda[2(A{:}\bar{D})Q - \bar{D}Q - Q\bar{D}] + n|\bar{D}|Q = 0. \tag{5.21}$$

Now take the L^2-inner product of Eq. (5.21) with Q: this gives

$$\frac{1}{2}\frac{d}{dt}\|Q\|^2 + (Q\bar{W} - \bar{W}Q, Q) + \lambda[2(A{:}\bar{D})(Q, A) - ((\bar{D}Q + Q\bar{D}), Q)] + n(|\bar{D}|Q, Q) = 0. \tag{5.22}$$

Some of the terms in Eq. (5.22) are now simplified; firstly,

$$(\boldsymbol{QW}-\boldsymbol{WQ},\boldsymbol{Q})=\int_\Omega(\boldsymbol{QW}:\boldsymbol{Q}-\boldsymbol{WQ}:\boldsymbol{Q})\mathrm{d}x$$
$$=\int_\Omega(\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{Q}:\boldsymbol{W}-\boldsymbol{QQ}^{\mathrm{T}}:\boldsymbol{W})\mathrm{d}x=0$$

using the identity $\boldsymbol{AB}:\boldsymbol{C}=\boldsymbol{B}:\boldsymbol{A}^{\mathrm{T}}\boldsymbol{C}=\boldsymbol{A}:\boldsymbol{CB}^{\mathrm{T}}$ and the skew-symmetry of $\boldsymbol{Q}$. In the same way we get

$$((\bar{\boldsymbol{D}}\boldsymbol{Q}+\boldsymbol{Q}\bar{\boldsymbol{D}}),\boldsymbol{Q})=-2\int_\Omega\boldsymbol{Q}^2:\bar{\boldsymbol{D}}\,\mathrm{d}x\le 2\|\boldsymbol{Q}\|^2\|_{L^2}\|\bar{\boldsymbol{v}}\|_{H^3},$$

and

$$(|\bar{\boldsymbol{D}}|\boldsymbol{Q},\boldsymbol{Q})\le\|\boldsymbol{Q}\|^2\|_{L^2}\|\bar{\boldsymbol{v}}\|_{H^3}.$$

Using these results, we arrive at the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{Q}\|^2-2K\|\boldsymbol{Q}\|^2\|_{L^2}\|\bar{\boldsymbol{v}}\|_{H^3}\le 0. \tag{5.23}$$

Since $\bar{\boldsymbol{v}}$ satisfies Eq. (5.18) we may apply Gronwall's lemma to conclude that $\boldsymbol{Q}=0$, a.e. in Ω_{T^*}; this establishes (a).

To show that (b) holds, we take the trace of both sides of Eq. $(5.2)_2$. Recalling that $\mathrm{tr}\,\bar{\boldsymbol{A}}=\mathrm{tr}\,\boldsymbol{A}_0=0$, and setting $Z=\mathrm{tr}\,\boldsymbol{A}$, we obtain

$$Z'+\bar{\boldsymbol{v}}\nabla Z-n|\bar{\boldsymbol{D}}|Z=0,$$
$$Z(0)=0. \tag{5.24}$$

Clearly, $Z(t)=0$ solves Eq. (5.24) and, by reasoning as has been done previously, it may be shown that Z is the only solution to Eq. (5.24). Thus (b) is established and the lemma is proven.

6. Existence and uniqueness for problem $\mathscr{P}$

The objective of this section is to show the existence and uniqueness for Problem $\mathscr{P}$. The former is obtained by a simple application of Schauder's fixed point theorem together with the results established in the preceding section. To this end, for $T>0$ we define the set R_T by

$$R_T=\{(\bar{\boldsymbol{v}},\bar{\boldsymbol{A}}),\bar{\boldsymbol{v}}\in C([0,T];D(\mathscr{L}))\cap L^2(0,T;\boldsymbol{H}^3),\bar{\boldsymbol{v}}'\in C([0,T];H)\cap L^2(0,T;V),$$
$$\bar{\boldsymbol{A}}\in L^\infty([0,T];IH^2),\bar{\boldsymbol{A}}'\in L^\infty([0,T];IH^1),\bar{\boldsymbol{v}}(0)=\boldsymbol{v}_0,\bar{\boldsymbol{A}}(0)=\boldsymbol{A}_0\ \text{in}\ \Omega,$$
$$\|\bar{\boldsymbol{v}}\|_{L^\infty(0,T;D(\mathscr{L}))\cap L^2(0,T;H^3)}+\|\bar{\boldsymbol{v}}'\|_{L^\infty(0,T;H)\cap L^2(0,T;V)}\le B_1,$$
$$\|\bar{\boldsymbol{A}}\|_{L^\infty(0,T;IH^2)}\le B_2,\|\bar{\boldsymbol{A}}'\|_{L^\infty(0,T;IH^1)}\le B_3\}$$

where B_1, B_2 and B_3 are defined in Lemma 3. As in [12], one can show that $R_T\ne\emptyset$, for all $T>0$, whenever

$$B_1\ge 2(C\|\boldsymbol{v}_0\|_{H^2}+\|\boldsymbol{A}_0\|_{H^2})\equiv\mathscr{D}_1 \tag{6.1}$$

with C being the constant in the estimate in Lemma 2. In what follows, we shall choose B_1 in such a way that it satisfies the restriction Eq. (6.1). We next set

$$\chi_T = C([0,T];V) \times C([0,T];IH^1).$$

Clearly, R_T is a closed convex subset of χ_T.

Now consider the map

$$\Phi : (\bar{\boldsymbol{v}}, \bar{\boldsymbol{A}}) \in R_T \subset \chi_T \to (\boldsymbol{v}, \boldsymbol{A})$$

where $\boldsymbol{v}$, $\boldsymbol{A}$ are the unique solutions to Eq. (5.1) and Eq. (5.2). We show that Φ maps R_T into itself, for T sufficiently small and for a suitable choice of B_1.

From Eq. (4.11) and the Sobolev inequality Eq. (5.10), it easily follows that for $(\bar{\boldsymbol{v}}, \bar{\boldsymbol{A}}) \in R_T$,

$$\begin{aligned}
&\|\mathrm{div}\,\tilde{\boldsymbol{S}}\|_{L^2(0,\,T;\,H^1)} \le \mathscr{Q}_1(B_2)\|\boldsymbol{v}\|_{L^2(0,\,T;\,H^3)} \le \mathscr{Q}_1(B_2)B_1,\\
&\|\mathrm{div}\,\tilde{\boldsymbol{S}}(0)\|_{L^2} \le C_1(1+\|\boldsymbol{A}_0\|_{H^2})\|\boldsymbol{A}_0\|_{H^2}\|\boldsymbol{v}_0\|_{H^2},\\
&\|\mathrm{div}\,\tilde{\boldsymbol{S}}'\|_{L^2(0,\,T;\,H^{-1})} \le C_2(1+B_2)B_3\|\bar{\boldsymbol{v}}\|_{L^2(0,T;H^2)} + \mathscr{Q}_3(B_2)\|\bar{\boldsymbol{v}}'\|_{L^2(0,\,T;\,H^1)}\\
&\qquad\qquad\qquad\qquad \le [C_2(1+B_2)B_3 + \mathscr{Q}_3(B_2)]B_1,
\end{aligned}$$

where $\mathscr{Q}_i(B_2)$, $i = 1\ldots3$, are polynomials homogeneous in B_2. Consequently, recalling the definition Eq. (5.4) of $\boldsymbol{F}$, we find that there exist constants $K_i > 0$ $(i = 1, 2, 3)$ such that

$$\begin{aligned}
&\|\boldsymbol{F}\|_{L^2(0,\,T;\,H^1)} \le K_1(\mathscr{Q}_1(B_2)+B_1)B_1 + \sqrt{n}\|\boldsymbol{b}\|_{L^2(0,\,T;\,H^1)} \equiv F_1B_1 + \mathscr{D}_2,\\
&\|\boldsymbol{F}(0)\|_{L^2} \le K_1\{(1+\|\boldsymbol{A}_0\|_{H^2})\|\boldsymbol{A}_0\|_{H^2}\|\boldsymbol{v}_0\|_{H^2} + \|\boldsymbol{v}_0\|^2_{H^2}\} + \sqrt{n}\|\boldsymbol{b}(0)\|_{L^2} \equiv \mathscr{D}_3,\\
&\|\boldsymbol{F}'\|_{L^2(0,\,T;\,H^{-1})} \le K_1[C_2(1+B_2)B_3 + \mathscr{Q}_3(B_2) + B_1]B_1 + \sqrt{n}\|\boldsymbol{b}'\|_{L^2(0,\,T;\,H^{-1})} \equiv F_2B_1 + \mathscr{D}_4,
\end{aligned} \tag{6.2}$$

in which $F_1 = K_1(\mathscr{Q}_1(B_2) + B_1)$ and $F_2 = K_1[(1+B_2)B_3 + \mathscr{Q}_3(B_2) + B_1]$. Taking

$$B_1 = \mathscr{D}_1 + 4C\sum_{i=2}^{4}\mathscr{D} \tag{6.3}$$

we have that Eq. (6.1) is satisfied. Moreover, from Eqs. (6.1) and (6.2) it follows that

$$\begin{aligned}
&C(\|\boldsymbol{v}_0\|_{H^2} + \|\boldsymbol{F}\|_{L^2(0,\,T;\,H^1)} + \|\boldsymbol{F}(0)\|_{L^2} + \|\boldsymbol{F}'\|_{L^2(0,\,T;\,H^{-1})})\\
&\le \tfrac{1}{2}B_1 + C(F_1+F_2)B_1 + C(\mathscr{D}_2+\mathscr{D}_3+\mathscr{D}_4)\\
&\le \tfrac{1}{2}B_1 + \tfrac{1}{4}B_1 + C(F_1+F_2)B_1 = (\tfrac{3}{4} + C(F_1+F_2))B_1.
\end{aligned} \tag{6.4}$$

Choosing

$$B_2 = 4B_1,$$

from Eq. (6.1) we at once verify that Eq. (5.7) is satisfied. Moreover, recalling that $B_3 = B_1\mathscr{R}(B_2)$ with $\mathscr{R}(B_2)$ a polynomial in B_2, we conclude that F_1 and F_2 defined in Eq. (6.2) become homogeneous polynomials in B_1. Thus, in view of Eq. (6.3), F_1 and F_2 can be increased by homogeneous polynomials in $\mathscr{D}$, where

$$\mathscr{D} = \sum_{i=1}^{4}\mathscr{D}_i.$$

Therefore, if we take the data so small as to satisfy the condition

$$C(F_1+F_2)<\tfrac{1}{4},$$

then from Eq. (6.4), Lemma 2 and Lemma 3, it may be deduced that Φ maps R_{T^*} into itself, with T^* being defined as in Lemma 3. Moreover, by the Ascoli-Arzelà theorem, R_{T^*} is compact in χ_{T^*}, and this together with the Schauder fixed point theorem gives the following result.

Theorem 4. *Assume that Ω has a* C^3 boundary, $\boldsymbol{b}\in L^2_{\rm loc}(R^+;\boldsymbol{H}^1)$, $\boldsymbol{b}'\in L^2_{\rm loc}(R^+;\boldsymbol{H}^{-1})$, $\boldsymbol{v}_0\in D(\mathscr{L})$, and $\boldsymbol{A}_0\in IH^2(\Omega)$. *Then there exist positive constants K and T such that if*

$$\|\boldsymbol{b}'\|_{L^2(0,T;H^1)\cap L^2(0,T;H^{-1})}+\|\boldsymbol{v}_0\|_{D(\mathscr{L})}+\|\boldsymbol{A}_0\|_{IH^2}\le K,$$

then Problem $\mathscr{P}$ admits at least one solution $(\boldsymbol{v},p,\boldsymbol{A})$ in $\Omega\times(0,T)$ with

$$\boldsymbol{v}\in L^2(0,T;\boldsymbol{H}^3),\qquad \boldsymbol{v}'\in L^2(0,T;V),\qquad p\in L^2(0,T;\boldsymbol{H}^2),$$

$$\boldsymbol{A}\in L^\infty(0,T;IH^2),\qquad \boldsymbol{A}'\in L^\infty(0,T;IH^1).$$

The constant K depends on Ω and on the material constants, while T depends on the data, the material constants and on Ω. Finally, the solution satisfies the estimate

$$\|\boldsymbol{v}\|_{L^2(0,T;H^3)\cap L^\infty(0,T;D(\mathscr{L}))}+\|\boldsymbol{v}'\|_{L^2(0,T;V)\cap L^\infty(0,T;H)}$$
$$+\|p\|_{L^2(0,T;H^2)}+\|\boldsymbol{A}\|_{L^\infty(0,T;IH^2)}+\|\boldsymbol{A}'\|_{L^\infty(0,T;IH^1)}\le\mathscr{H},$$

where $\mathscr{H}$ depends on the data in such a way that

$$\mathscr{H}\to0\ \text{ as }\ \|\boldsymbol{b}\|_{L^2(0,T;H^1)}+\|\boldsymbol{b}'\|_{L^2(0,T;H^{-1})}+\|\boldsymbol{v}_0\|_{H^2}+\|\boldsymbol{A}_0\|_{IH^2}\to0.$$

6.1. Uniqueness of the solution

Let $(\boldsymbol{v}_1,p_1,\boldsymbol{A}_1)$, $(\boldsymbol{v}_2,p_2,\boldsymbol{A}_2)$ be two solutions corresponding to the same data. Now Theorem 4 yields the estimate

$$\|\boldsymbol{A}_1\|_{L^\infty(0,T;H^2)}+\|\boldsymbol{A}_2\|_{L^\infty(0,T;H^2)}\le\delta,\tag{6.5}$$

where δ can be taken sufficiently small if the body force and the initial data are sufficiently small in suitable norms. Setting

$$\boldsymbol{v}=\boldsymbol{v}_1-\boldsymbol{v}_2,\ p=p_1-p_2,\ \boldsymbol{A}=\boldsymbol{A}_1-\boldsymbol{A}_2,$$

$$\boldsymbol{S}_1=\tilde{\boldsymbol{S}}(\boldsymbol{A}_1,\boldsymbol{D}_1),\ \boldsymbol{S}_2=\tilde{\boldsymbol{S}}(\boldsymbol{A}_2,\boldsymbol{D}_2),\ G_1=G(\boldsymbol{A}_1,\boldsymbol{W}_1,\boldsymbol{D}_1),\ G_2=G(\boldsymbol{A}_2,\boldsymbol{W}_2,\boldsymbol{D}_2),$$

we obtain from Eqs. (4.10), (4.11), (4.12) and (4.13) the pair of equations

$$\mathrm{Re}[\boldsymbol{v}'+b(\boldsymbol{v},\boldsymbol{v})+b(\boldsymbol{v}_1,\boldsymbol{v})+b(\boldsymbol{v},\boldsymbol{v}_1)]+\mathscr{L}\boldsymbol{v}=P\ \mathrm{div}(\boldsymbol{S}_1-\boldsymbol{S}_2),$$

$$\boldsymbol{A}'+B(\boldsymbol{v},\boldsymbol{A})+B(\boldsymbol{v}_1,\boldsymbol{A})+B(\boldsymbol{v},\boldsymbol{A}_1)=G_1-G_2.\tag{6.6}$$

Taking the L^2-inner product of Eq. $(6.6)_1$ with $\boldsymbol{v}$ and of Eq. $(6.6)_2$ with $\boldsymbol{A}$, and integrating, we find that

$$\mathrm{Re}\left[\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{v}\|^2_{L^2}+(b(\boldsymbol{v},\boldsymbol{v}_1),\boldsymbol{v})\right]+|\boldsymbol{v}|^2_{H^1}=((\boldsymbol{S}_1-\boldsymbol{S}_2),\boldsymbol{D}),$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{A}\|^2_{L^2}+(B(\boldsymbol{v},\boldsymbol{A}_1),\boldsymbol{A})=((G_1-G_2),\boldsymbol{A}), \tag{6.7}$$

where, in order to simplify the notation, we set $(\cdot,\cdot)_{L^2}\equiv(\cdot,\cdot)$. Using Eqs. (5.11) and (6.5) it follows that

$$|(b(\boldsymbol{v},\boldsymbol{v}_1),\boldsymbol{v})|\le c\|\boldsymbol{v}_1\|_{H^3}\|\boldsymbol{v}\|^2_{L^2},$$
$$|(B(\boldsymbol{v},\boldsymbol{A}_1),\boldsymbol{A})|\le c\delta\|\boldsymbol{v}\|_{H^1}\|\boldsymbol{A}\|_{L^2}.$$

With the help of the Cauchy-Schwarz inequality, we obtain from Eq. (6.6)

$$\mathrm{Re}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{v}\|^2_{L^2}+|\boldsymbol{v}|^2_{H^1}\le c\mathrm{Re}\|\boldsymbol{v}_1\|_{H^3}\|\boldsymbol{v}\|^2_{L^2}+\|\boldsymbol{S}_1-\boldsymbol{S}_2\|_{L^2}\|\boldsymbol{v}\|_{H^1},$$
$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{A}\|^2_{L^2}\le c\delta\|\boldsymbol{v}\|_{H^1}\|\boldsymbol{A}\|_{L^2}+\|G_1-G_2\|_{L^2}\|\boldsymbol{A}\|_{L^2}. \tag{6.8}$$

We now need to establish some estimates for the quantities $\boldsymbol{S}_1-\boldsymbol{S}_2$ and G_1-G_2. To this end we observe, using Eqs. (2.19) and $(4.6)_3$, that

$$\boldsymbol{S}_1-\boldsymbol{S}_2=\gamma\left\{N_\mathrm{p}\left[\frac{1}{n}\boldsymbol{I}(\boldsymbol{A}:\boldsymbol{D}_1+\boldsymbol{A}_2:\boldsymbol{D})+(\boldsymbol{A}_1:\boldsymbol{D}_1)\boldsymbol{A}_1-(\boldsymbol{A}_2:\boldsymbol{D}_2)\boldsymbol{A}_2\right]\right.$$
$$\left.+N_\mathrm{s}(\boldsymbol{D}_1\boldsymbol{A}+\boldsymbol{A}\boldsymbol{D}_1+\boldsymbol{D}\boldsymbol{A}_2+\boldsymbol{A}_2\boldsymbol{D})\right\}. \tag{6.9}$$

From Eqs. (5.10) and (6.5) we immediately find that

$$\|\boldsymbol{D}_1\boldsymbol{A}+\boldsymbol{A}\boldsymbol{D}_1+\boldsymbol{D}\boldsymbol{A}_2+\boldsymbol{A}_2\boldsymbol{D}\|_{L^2}\le c(\|\boldsymbol{v}_1\|_{H^3}\|\boldsymbol{A}\|_{L^2}+\delta\|\boldsymbol{v}\|_{H^1}). \tag{6.10}$$

Moreover, again by Eqs. (5.10) and (6.5) it follows that

$$\|(\boldsymbol{A}_1:\boldsymbol{D}_1)\boldsymbol{A}_1-(\boldsymbol{A}_2:\boldsymbol{D}_2)\boldsymbol{A}_2\|_{L^2}=\|\boldsymbol{A}\otimes\boldsymbol{A}_1:\boldsymbol{D}_1+\boldsymbol{A}_2\otimes\boldsymbol{A}:\boldsymbol{D}_2+\boldsymbol{A}_2\otimes\boldsymbol{A}_1:\boldsymbol{D}\|_{L^2}$$
$$\le c\delta(\|\boldsymbol{v}_1\|_{H^3}+\|\boldsymbol{v}_2\|_{H^3})\|\boldsymbol{A}\|_{L^2}+\delta^2\|\boldsymbol{v}\|_{H^1}, \tag{6.11}$$

in which $\boldsymbol{D}=\boldsymbol{D}_1-\boldsymbol{D}_2$.

In the same way, we can show that

$$\|(\boldsymbol{A}_i:\boldsymbol{D})\|_{L^\infty}\le c\delta\|\boldsymbol{v}_i\|_{H^3},\quad i=1,2. \tag{6.12}$$

We thus conclude, using Eqs. (6.5), (6.9), (6.10), (6.11) and (6.12), that

$$\|\boldsymbol{S}_1-\boldsymbol{S}_2\|_{L^2}\le\mathscr{R}(\delta)[(\|\boldsymbol{v}_1\|_{H^3}+\|\boldsymbol{v}_2\|_{H^3})\|\boldsymbol{A}\|_{H^2}+\|\boldsymbol{v}\|_{H^1}] \tag{6.13}$$

where $\mathscr{R}(\delta)$ is a homogeneous polynomial in δ. Substituting Eq. (6.13) in Eq. $(6.8)_1$, setting

$$V=\|\boldsymbol{v}_1\|_{H^3}+\|\boldsymbol{v}_2\|_{H^3}, \tag{6.14}$$

and using the Cauchy–Schwarz inequality, we find that

$$\mathrm{Re}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{v}\|^2_{L^2} + |\boldsymbol{v}|^2_{H^1} \le \frac{3\mathscr{R}(\delta)}{2}\|\boldsymbol{v}\|^2_{H^1} + \frac{1}{2}\mathscr{R}(\delta)V^2\|\boldsymbol{A}\|^2_{H^2} + c\ \mathrm{Re}\ V\|\boldsymbol{v}\|^2_{L^2}. \tag{6.15}$$

By the Poincaré–Friedrichs inequality Eq. (3.11) we have

$$|\boldsymbol{v}|_{H^1} \ge \frac{1}{2}(1+c_0)\|\boldsymbol{v}\|_{H^1}$$

and so, taking δ sufficiently small so that

$$\eta \equiv \frac{1}{2}(1+c_0) - 3\mathscr{R}(\delta)/2 > 0$$

we have, from Eq. (6.15),

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{v}\|^2_{L^2} \le c(V^2\|\boldsymbol{A}\|^2_{H^2} + V\|\boldsymbol{v}\|^2_{L^2}) - \frac{2\eta}{\mathrm{Re}}\|\boldsymbol{v}\|^2_{H^1}. \tag{6.16}$$

We now need to estimate $\|G_1 - G_2\|_{L^2}$. To this end, we observe that

$$G_1 - G_2 = -(\boldsymbol{W}_1\boldsymbol{A} - \boldsymbol{A}_1\boldsymbol{W} + \boldsymbol{W}\boldsymbol{A}_2 - \boldsymbol{A}\boldsymbol{W}_2) - \lambda\left(\boldsymbol{D}_1\boldsymbol{A} + \boldsymbol{A}_1\boldsymbol{D} + \boldsymbol{D}\boldsymbol{A}_2 + \boldsymbol{A}\boldsymbol{D}_2 + \frac{2}{n}\boldsymbol{D}\right)$$
$$- n(|\boldsymbol{D}_1|\boldsymbol{A} + (|\boldsymbol{D}_1| - |\boldsymbol{D}_2|)\boldsymbol{A}_2).$$

By a sequence of arguments analogous to those contained in Eqs. (6.10), (6.11), (6.12) and (6.13), we conclude that

$$\|G_1 - G_2\|_{L^2} \le C(V\|\boldsymbol{A}\|_{L^2} + \|\boldsymbol{v}\|_{H^1}),$$

and by using this relation in Eq. $(6.8)_2$ together with the Cauchy-Schwarz inequality, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{A}\|^2_{L^2} \le c[\|\boldsymbol{v}\|^2_{H^1} + (V+1)\|\boldsymbol{A}\|^2_{L^2}]. \tag{6.17}$$

If we multiply Eq. (6.17) by $2\eta/c\mathrm{Re}$ and add the resulting equation to Eq. (6.16), we find that

$$\frac{\mathrm{d}E}{\mathrm{d}t} \le c(V^2 + V + 1)E,$$

where $E = \|\boldsymbol{v}\|^2_{L^2} + \|\boldsymbol{A}\|^2_{L^2}$. Therefore, recalling that $V \in L^2\ (0,\ T)$, it follows from Gronwall's lemma that $E(t) = 0$ a.e. in $[0,\ T]$. We have thus proved the following.

Theorem 5. *There exists a positive constant* K such that if

$$\|\boldsymbol{b}'\|_{L^2(0,\,T;\,H^1)\cap L^2(0,\,T;\,H^{-1})} + \|\boldsymbol{v}_0\|_{D(\mathscr{L})} + \|\boldsymbol{A}_0\|_{IH^2} \le K,$$

then Problem $\mathscr{P}$ admits at most one solution in the class of solutions ($\boldsymbol{v}$, p, $\boldsymbol{A}$) such that[2]

$$\boldsymbol{v}\in L^2(0, T; \boldsymbol{H}^3), \quad \boldsymbol{v}'\in L^2(0, T; V), \quad \nabla p\in L^2(0, T; \boldsymbol{H}^1),$$
$$\boldsymbol{A}\in L^\infty(0, T; IH^2), \quad \boldsymbol{A}'\in L^\infty(0, T; IH^1).$$

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