CONVERGENCE IN THE INCOMPRESSIBLE LIMIT OF FINITE ELEMENT APPROXIMATIONS BASED ON THE HU-WASHIZU FORMULATION

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Abstract. The classical Hu-Washizu mixed formulation for plane problems in elasticity is examined afresh, with the emphasis on behavior in the incompressible limit. The continuous problem is shown to be uniformly well-posed in the incompressible limit, by recourse to an equivalent modified formulation. For the discrete problems, conditions for uniform convergence are made explicit, and these conditions are shown to be met by particular choices of approximations based on quadrilateral elements. These choices include bases that are well known, as well as newly constructed bases. The modified Hu-Washizu formulation is shown to be stable and convergent in circumstances in which the classical formulation will not exhibit these properties. The theoretical results are explored further through selected numerical examples.

Key words. Hu-Washizu, mixed finite elements, low-order approximations, uniform convergence

AMS subject classifications. 65N30, 65N15, 74B10

1. Introduction. There now exists a large number of publications devoted to the task of constructing and analyzing finite element approximations for problems in solid mechanics, in which it is necessary to circumvent volumetric locking, or alternatively expressed, to ensure uniform convergence in the incompressible limit. A further consideration is that of obtaining approximations of high quality using low-order elements, and for coarse meshes: when quadrilateral elements are used in two-dimensional problems, and hexahedral in three, it is well-known that the standard piecewise bi- and trilinear approximations in two and three dimensions lead to poor approximations when coarse meshes are used.

There is a growing literature dealing with the well-posedness of stabilization methods, and Brezzi and Fortin [9] have undertaken a detailed abstract analysis that is applicable to a wide range of such approaches. Methods associated with the enrichment or enhancement of the strain or stress field by the addition of carefully chosen basis functions have proved to be highly effective and popular. The key work dealing with enhanced assumed strain formulations is that of Simo and Rifai [22]. Reddy and Simo [21] have carried out a detailed analysis of the convergence of enhanced assumed strain methods, for affine-equivalent meshes, and for the compressible and incompressible cases. They established an a priori error estimate for displacements that confirms convergence at the standard linear rate. Braess [5] has re-examined the sufficient conditions for convergence, in particular relating the stability condition to a strengthened Cauchy inequality, and elucidating the influence of the Lamé constant λ. The case of limiting compressibility has been the subject of a recent analysis by Braess, Carstensen and Reddy [6], in which λ-independent asymptotic convergence of the displacement error is obtained, for a class of meshes. The assumed

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stress approach [20] leads to a formulation very similar to that based on enhanced strains, and in fact the two are equivalent under certain conditions [1, 6].

The purpose of this work is to focus on three-field mixed formulations as an approach to overcoming the problems encountered in standard formulations, for problems in elasticity. The starting point is the Hu-Washizu formulation [16, 24], in which the unknown variables are displacement, strain, and stress. This formulation, which was in fact first introduced by Fraeijs de Veubeke [12] (see [13] and [14], a reprint of [13], and the historical comments in [11]), serves as the point of departure for the development of enhanced strain formulations.

The structure of the rest of this work is as follows. Section 2 is devoted to the continuous Hu-Washizu problem, in its original and modified settings, and to the presentation of relevant results on well-posedness. In Section 3 the discrete formulations, based on finite element approximations, are presented. Section 4 focuses on the analysis of the modified formulation, and on the conditions for its uniform well-posedness in the incompressible limit. For this purpose it is necessary to introduce discrete deviatoric and spherical operators on the spaces of discrete stresses and strains. An abstract requirement for well-posedness is then that the trace of the space of discrete spherical stresses forms, with the space of displacements, a stable pairing in the sense of the classical Stokes problem. Of exclusive interest here are situations corresponding to low-order approximations, with these being built on a space of displacements corresponding to piecewise continuous bilinear approximations on quadrilaterals. As with the Stokes problem, in which the pressure space of piecewise constants has to be modified to extract from it a space of so-called checkerboard modes, it can be shown that the lack of stability resulting from the presence of a checkerboard mode is confined to the stress, and does not affect the displacement. This result is important in the computational context, especially in situations in which the displacement is the primary variable of interest. In Section 5, an a priori error estimate for the displacement is derived, making clear that the finite element solution converges with optimal order to the exact solution uniformly in the incompressible limit. Similarly in Section 6, an a priori error estimate for the postprocessed stress is presented. Finally, in Section 7, the results of Section 5 are explored through two numerical examples. The numerical results reflect the good performance of the modified formulation.

2. The boundary value problem of elasticity. In the context of elasticity, vector- and tensor- or matrix-valued functions will be written in boldface form. The scalar product of two tensors or matrices $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ will be denoted by $\boldsymbol{\sigma} : \boldsymbol{\tau}$, and is given by $\boldsymbol{a} : \boldsymbol{b} = a_{ij} b_{ij}$, the summation convention on repeated indices being invoked.

Consider a homogeneous isotropic linear elastic material body which occupies a bounded domain Ω in $\mathbb{R}^2$ with Lipschitz boundary Γ. For a prescribed body force $\boldsymbol{f} \in L^2(\Omega)^2$, the governing equilibrium equation in Ω reads

$$-\operatorname{div} \boldsymbol{\sigma} = \boldsymbol{f} \ , \tag{2.1}$$

where $\boldsymbol{\sigma}$ is the symmetric Cauchy stress tensor. The infinitesimal strain tensor $\boldsymbol{d}$ is defined as a function of the displacement $\boldsymbol{u}$ by

$$\boldsymbol{d} = \boldsymbol{\varepsilon}(\boldsymbol{u}) := \tfrac{1}{2}(\nabla \boldsymbol{u} + [\nabla \boldsymbol{u}]^t) \ . \tag{2.2}$$

The displacement is assumed to satisfy the homogeneous Dirichlet boundary condition

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad \Gamma. \tag{2.3}$$

With the fourth-order elasticity tensor denoted by $\mathcal{C}$, the constitutive equation reads

$$\boldsymbol{\sigma} = \mathcal{C}\boldsymbol{d} := \lambda(\operatorname{tr}\boldsymbol{d})\mathbf{1} + 2\mu\,\boldsymbol{d}\ . \tag{2.4}$$

Here, $\mathbf{1}$ is the identity tensor, and λ and μ are the Lamé parameters, which are constant in view of the assumption of a homogeneous body, and which are assumed positive. Of particular interest is the incompressible limit, which corresponds to $\lambda \to \infty$.

The inverse $\mathcal{C}^{-1}$ of $\mathcal{C}$ is given by

$$\boldsymbol{d} = \mathcal{C}^{-1}\boldsymbol{\sigma} = \frac{1}{2\mu}\left(\boldsymbol{\sigma} - \gamma(\operatorname{tr}\boldsymbol{\sigma})\mathbf{1}\right), \quad \gamma := \frac{\lambda}{\kappa}, \quad \kappa := 2(\mu+\lambda)\ .$$

Standard weak formulation. We will make use of the space $L^2(\Omega)$ of square-integrable functions defined on Ω with the inner product and norm being denoted by $(\cdot,\cdot)_0$ and $\|\cdot\|_0$, respectively. We will also make use of the Sobolev spaces $H^m(\Omega)$, for nonnegative integers m. The space $H^1_0(\Omega)$ consists of functions in $H^1(\Omega)$ which vanish on the boundary in the sense of traces.

For the weak or variational formulations we will require the space $V := [H^1_0(\Omega)]^2$ of displacements; this is a Hilbert space with inner product $(\cdot,\cdot)_1$ and norm $\|\cdot\|_1$ defined in the standard way; that is, $(\boldsymbol{u},\boldsymbol{v})_1 := \sum_{i=1}^{2}(u_i,v_i)_1$, with the norm being induced by this inner product. The space of stresses is denoted by S, while the space of strains is denoted by D. For the continuous case these spaces are equal, and $D := \{\boldsymbol{e} \mid e_{ji} = e_{ij},\ e_{ij} \in L^2(\Omega)\} =: S$, with the norm $\|\cdot\|_0$ generated in the standard way by the L^2-norm. We also introduce the space S_0 defined by

$$S_0 := \{\boldsymbol{\tau} \in S \mid (\boldsymbol{\tau},\mathbf{1})_0 = 0\}\ ;$$

this is a closed subspace of S.

Remark 2.1. *It is crucial that the stresses be sought in S_0 rather than in S. A similar consideration appears in the conditions for well-posedness of the Hellinger-Reissner problem [6]. The subspace S_0 is, however, a natural subspace in which to work, since for the continuous problem the solution $(\boldsymbol{u},\boldsymbol{d},\boldsymbol{\sigma})$ satisfies $\operatorname{tr}\boldsymbol{\sigma} = \kappa\operatorname{div}\boldsymbol{u}$, and using the Dirichlet boundary condition it is seen that $\int_\Omega \operatorname{tr}\boldsymbol{\sigma}\ dx = 0$.*

Define the bilinear form $A(\cdot,\cdot)$ and linear functional $\ell(\cdot)$ by

$$\begin{array}{lrcl} A : V\times V \to \mathbb{R}, & A(\boldsymbol{u},\boldsymbol{v}) & := & \int_\Omega \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}):\boldsymbol{\varepsilon}(\boldsymbol{v})\ dx\ , \\ \ell : V \to \mathbb{R}, & \ell(\boldsymbol{v}) & := & \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\ dx\ . \end{array}$$

Then the standard form of the weak problem for elasticity is as follows: given $\ell \in V'$, find $\boldsymbol{u} \in V$ that satisfies

$$A(\boldsymbol{u},\boldsymbol{v}) = \ell(\boldsymbol{v})\ , \quad \boldsymbol{v} \in V\ . \tag{2.5}$$

The assumptions on $\mathcal{C}$ guarantee that $A(\cdot,\cdot)$ is symmetric, continuous, and V-elliptic.

It is known [7, 23] that (2.5) has a unique solution $\boldsymbol{u} \in [H^2(\Omega)]^2$ and that there exists a constant C, independent of λ, such that

$$\|\boldsymbol{u}\|_2 + \lambda\|\operatorname{div}\boldsymbol{u}\|_1 \le C\,\|\boldsymbol{f}\|_0\ . \tag{2.6}$$

Mixed formulations. The problem described above may be cast in a variety of alternative mixed forms, the term ‘mixed’ carrying in this context the connotation that the resulting weak formulation has a link to a saddlepoint problem. We focus on the Hu-Washizu formulation, in which the displacement, strain, and stress are unknown variables. The standard Hu-Washizu formulation is obtained by considering the constitutive equation, the strain-displacement equation and the equation of equilibrium in weak form. Here, we will consider a more general form depending on a parameter $\alpha := \alpha(\mu,\lambda) \le 1$: find $(\boldsymbol{u},\boldsymbol{d},\boldsymbol{\sigma}) \in V \times D \times S_0$ such that

$$\begin{array}{lclcll} a_\alpha((\boldsymbol{u},\boldsymbol{d}),(\boldsymbol{v},\boldsymbol{e})) & + & b_\alpha((\boldsymbol{v},\boldsymbol{e}),\boldsymbol{\sigma}) & = & \ell(\boldsymbol{v}) \ , & (\boldsymbol{v},\boldsymbol{e}) \in V \times D \ , \\ b_\alpha((\boldsymbol{u},\boldsymbol{d}),\boldsymbol{\tau}) & - & \frac{(1-\alpha)\gamma^2}{\lambda}\, c(\boldsymbol{\sigma},\boldsymbol{\tau}) & = & 0 \ , & \boldsymbol{\tau} \in S_0 \ , \end{array} \tag{2.7}$$

where the bilinear forms are defined by

$$\begin{aligned} a_\alpha((\boldsymbol{u},\boldsymbol{d}),(\boldsymbol{v},\boldsymbol{e})) &:= 2\mu(\boldsymbol{d},\boldsymbol{e})_0 + \alpha\lambda(\operatorname{tr}\boldsymbol{d},\operatorname{tr}\boldsymbol{e})_0 \ , \\ b_\alpha((\boldsymbol{v},\boldsymbol{e}),\boldsymbol{\sigma}) &:= (\boldsymbol{\varepsilon}(\boldsymbol{v}) - 2\mu\,\mathcal{C}^{-1}\boldsymbol{e},\boldsymbol{\sigma})_0 - \alpha\gamma(\operatorname{tr}\boldsymbol{\sigma},\operatorname{tr}\boldsymbol{e})_0 \ , \\ c(\boldsymbol{\sigma},\boldsymbol{\tau}) &:= (\operatorname{tr}\boldsymbol{\sigma},\operatorname{tr}\boldsymbol{\tau})_0 \ . \end{aligned}$$

The standard Hu-Washizu formulation is given by $\alpha = 1$. By using the Babuška-Brezzi theory [8, 15], it is easy to show that for $\alpha = 1$ there exists a unique solution. However in this case, the continuity constant of the bilinear form $a_1(\cdot,\cdot)$ tends to infinity as λ does. Thus, our analysis cannot be based directly on the mixed formulation with $\alpha = 1$.

In what follows we will make extensive use of the L^2-orthogonal decomposition of S into its deviatoric and spherical parts. Define the L^2-orthogonal projections sph and dev on S by $\operatorname{sph}\boldsymbol{\tau} := (1/2)\,(\operatorname{tr}\boldsymbol{\tau})\mathbf{1}$, and $\operatorname{dev}\boldsymbol{\tau} := \boldsymbol{\tau} - \operatorname{sph}\boldsymbol{\tau}$. We note that $\operatorname{dev} S$ is a proper subset of S_0.

Lemma 2.2. *For $\alpha \neq -\frac{\mu}{\lambda}$, there exists a unique solution $(\boldsymbol{u},\boldsymbol{d},\boldsymbol{\sigma}) \in V \times D \times S_0$ of the modified Hu-Washizu formulation (2.7). Moreover, the solution does not depend on α and satisfies the bound*

$$\|\boldsymbol{u}\|_1 + \|\boldsymbol{d}\|_0 + \|\boldsymbol{\sigma}\|_0 \le C\|\ell\|_{V'} \ ,$$

in which the constant C is independent of λ.

Proof. In a first step, we show that the solution of (2.7) does not depend on α. The first equation in (2.7) with $\boldsymbol{v} = 0$ yields

$$0 = (\mathcal{C}\boldsymbol{d} - \boldsymbol{\sigma}, \boldsymbol{e} + (\alpha-1)\gamma\operatorname{tr}\boldsymbol{e}\mathbf{1})_0 =: (\mathcal{C}\boldsymbol{d} - \boldsymbol{\sigma}, \mathcal{B}_\alpha\boldsymbol{e})_0, \quad \boldsymbol{e} \in D \ . \tag{2.8}$$

The inverse of $\mathcal{B}_\alpha$ exists for $\alpha \neq -\frac{\mu}{\lambda}$ and satisfies $\mathcal{B}_\alpha^{-1}\boldsymbol{e} = \boldsymbol{e} - \frac{(\alpha-1)\lambda}{2(\mu+\alpha\lambda)}\operatorname{tr}\boldsymbol{e}\mathbf{1}$. As a consequence, we find $\boldsymbol{\sigma} = \mathcal{C}\boldsymbol{d}$ and thus $\operatorname{tr}\boldsymbol{\sigma} = \kappa\operatorname{tr}\boldsymbol{d}$. The bilinear forms $a_\alpha(\cdot,\cdot)$ and $b_\alpha(\cdot,\cdot)$ can be equivalently written as

$$\begin{aligned} a_\alpha((\boldsymbol{w},\boldsymbol{g}),(\boldsymbol{v},\boldsymbol{e})) &= a_1((\boldsymbol{w},\boldsymbol{g}),(\boldsymbol{v},\boldsymbol{e})) + (\alpha-1)\lambda c(\boldsymbol{g},\boldsymbol{e}) \ , \\ b_\alpha((\boldsymbol{v},\boldsymbol{e}),\boldsymbol{\tau}) &= b_1((\boldsymbol{v},\boldsymbol{e}),\boldsymbol{\tau}) - (\alpha-1)\gamma c(\boldsymbol{\tau},\boldsymbol{e}) \ . \end{aligned} \tag{2.9}$$

Using $\lambda c(\boldsymbol{d},\boldsymbol{e}) = \gamma c(\boldsymbol{\sigma},\boldsymbol{e})$ and (2.9), it is trivial to see that the solution does not depend on α.

The proof of uniform stability is carried out by showing that the bilinear forms for $\alpha = 0$ satisfy the conditions for well-posedness for the extended case in which $c(\cdot,\cdot) \neq 0$ (see [8], Section II.1.2 and [4]). First, we note that $c(\cdot,\cdot)$ is symmetric and positive semi-definite, and that $\ker B^t := \{\boldsymbol{\tau} \in S_0 \mid b_0((\boldsymbol{v},\boldsymbol{e}),\boldsymbol{\tau}) = 0, \ \ (\boldsymbol{v},\boldsymbol{e}) \in V \times D\} = \{0\}$, so that $c(\cdot,\cdot)$ plays no further role in determining the well-posedness of the problem.

Next, it is easy to see that all bilinear forms for $\alpha = 0$ are continuous, and that the continuity constants do not depend on λ. It remains therefore to verify that the coercivity and inf-sup conditions are satisfied. We establish the coercivity of the bilinear form $a_0(\cdot,\cdot)$ on the kernel Z of $b_0(\cdot,\cdot)$, which is given by

$$\begin{aligned} Z &= \{(\boldsymbol{v},\boldsymbol{e}) \in V \times D \mid b_0((\boldsymbol{v},\boldsymbol{e}),\boldsymbol{\tau}) = 0,\ \boldsymbol{\tau} \in S_0\} \\ &= \{(\boldsymbol{v},\boldsymbol{e}) \in V \times D \mid \boldsymbol{\varepsilon}(\boldsymbol{v}) - 2\mu\mathcal{C}^{-1}\boldsymbol{e} \in S_0^{\perp}\}\ . \end{aligned}$$

Let $(\boldsymbol{v},\boldsymbol{e})$ be in Z; using Korn's inequality and the uniform continuity of $\mathcal{C}^{-1}$, and noting that $(\boldsymbol{\varepsilon}(\boldsymbol{v}),\mathbf{1})_0 = 0$, we find that

$$a_0((\boldsymbol{v},\boldsymbol{e}),(\boldsymbol{v},\boldsymbol{e})) = 2\mu\|\boldsymbol{e}\|_0^2 \geq c(\|\boldsymbol{e}\|_0^2 + \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_0^2) \geq c(\|\boldsymbol{e}\|_0^2 + \|\boldsymbol{v}\|_1^2)\ .$$

We note that the coercivity constant depends on μ but not on λ. Next, to establish the inf-sup condition we note that there exists a constant $C < \infty$ such that for each $q \in M := \{q \in L^2(\Omega) \mid \int_\Omega q\ dx = 0\}$, there exists $\boldsymbol{v}_q \in V$ satisfying

$$\operatorname{div}\boldsymbol{v}_q = q, \quad \|\boldsymbol{v}_q\|_1 \leq C\|q\|_0\ , \tag{2.10}$$

(see, for example, [15]). For $\boldsymbol{\tau} \in S_0$, we have $\operatorname{tr}\boldsymbol{\tau} \in M$, and we define $\boldsymbol{e}_{\boldsymbol{\tau}} := \operatorname{dev}(\boldsymbol{\varepsilon}(\boldsymbol{v}_{\operatorname{tr}\boldsymbol{\tau}}) - \boldsymbol{\tau}) \in D$. Then using (2.10), the norm of $(\boldsymbol{v}_{\operatorname{tr}\boldsymbol{\tau}},\boldsymbol{e}_{\boldsymbol{\tau}})$ is bounded by

$$\|\boldsymbol{v}_{\operatorname{tr}\boldsymbol{\tau}}\|_1^2 + \|\boldsymbol{e}_{\boldsymbol{\tau}}\|_0^2 \leq C(\|\operatorname{tr}\boldsymbol{\tau}\|_0^2 + \|\operatorname{dev}\boldsymbol{\tau}\|_0^2) \leq C\|\boldsymbol{\tau}\|_0^2\ .$$

The inf-sup condition now results from the orthogonality of the decomposition of $\boldsymbol{\tau}$ into its spherical and deviatoric parts; using also (2.10), we find that

$$\begin{aligned} b_0((\boldsymbol{v}_{\operatorname{tr}\boldsymbol{\tau}},\boldsymbol{e}_{\boldsymbol{\tau}}),\boldsymbol{\tau}) &= (\operatorname{sph}\boldsymbol{\varepsilon}(\boldsymbol{v}_{\operatorname{tr}\boldsymbol{\tau}}),\operatorname{sph}\boldsymbol{\tau})_0 + (\operatorname{dev}\boldsymbol{\tau},\operatorname{dev}\boldsymbol{\tau})_0 \\ &= (1/2)(\operatorname{div}\boldsymbol{v}_{\operatorname{tr}\boldsymbol{\tau}},\operatorname{tr}\boldsymbol{\tau})_0 + (\operatorname{dev}\boldsymbol{\tau},\operatorname{dev}\boldsymbol{\tau})_0 \\ &= (1/2)\|\operatorname{tr}\boldsymbol{\tau}\|_0^2 + \|\operatorname{dev}\boldsymbol{\tau}\|_0^2 = \|\boldsymbol{\tau}\|_0^2\ . \end{aligned}$$

□

REMARK 2.3. *The result of Lemma 2.2 can be generalized to the case* $\alpha = -\frac{\mu}{\lambda}$ *for which the operator* $\mathcal{B}_\alpha$ *is singular. The kernel of* $\mathcal{B}_\alpha$ *is given by* $\{\boldsymbol{e} \in D \mid \operatorname{dev}\boldsymbol{e} = 0\}$*. To obtain a unique solution of (2.7), we therefore have to seek a solution in the subspace* $\{(\boldsymbol{v},\boldsymbol{e},\boldsymbol{\tau}) \in V \times D \times S_0 \mid \operatorname{tr}\boldsymbol{\tau} = \kappa\operatorname{tr}\boldsymbol{e}\}$*.*

3. Finite element formulations. Let $\mathcal{T}_h$ be a quasi-uniform, shape-regular quadrilateral triangulation of the polygonal domain Ω. The diameter of an element K in $\mathcal{T}_h$ is denoted by h_K. Finite element spaces are defined by maps from a reference square $\hat{K} = (-1,1)^2$. For nonnegative integer k, let $\mathcal{P}_k(\cdot)$ denote the space of polynomials in two variables of total degree less than or equal to k, and $\mathcal{Q}_k(\cdot)$ the space of polynomials in two variables of total degree less than or equal to k in each variable. A typical element $K \in \mathcal{T}_h$ is generated by an isoparametric map F_K from the reference element $\hat{K}$. It is clear that if $\hat{v} \in \mathcal{Q}_1(\hat{K})$, then $\hat{v} \circ F_K^{-1}$ is in general not a polynomial on the quadrilateral K.

The finite element space for the displacement is taken to be the space of continuous functions whose restrictions to an element K are obtained by maps of bilinear functions from the reference element; that is,

$$V_h := \left\{\boldsymbol{v}_h \in V,\ \boldsymbol{v}_h|_K = \hat{\boldsymbol{v}}_h \circ F_K^{-1}, \hat{\boldsymbol{v}}_h \in \mathcal{Q}_1(\hat{K})^2 \ \text{ for all } \ K \in \mathcal{T}_h\right\}\ .$$

The spaces of stresses and strains are discretized by defining the finite-dimensional spaces

$$S_h := \left\{ \boldsymbol{\tau}_h \in S_0 \mid (\boldsymbol{\tau}_h|_K)_{ij} = (\hat{\boldsymbol{\tau}}_h)_{ij} \circ F_K^{-1}, \;\; \hat{\boldsymbol{\tau}}_h \in S_\square \text{ for all } K \in \mathcal{T}_h \right\} ,$$

$$D_h := \left\{ \boldsymbol{e}_h \in S_0 \mid (\boldsymbol{e}_h|_K)_{ij} = (\hat{\boldsymbol{e}}_h)_{ij} \circ F_K^{-1}, \;\; \hat{\boldsymbol{e}}_h \in D_\square \text{ for all } K \in \mathcal{T}_h \right\} ,$$

where $D_\square$ and $S_\square$ are the reference bases of strains and stresses, defined on $\hat{K}$. These two variables are defined locally on each element and no continuity conditions apply at the element boundaries. Moreover, we define the space M_h by

$$M_h := \operatorname{tr} S_h \ .$$

Before giving some concrete examples, we recall the Voigt representation of the tensorial quantities stress and strain in vectorial form, in two dimensions. These are written as

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} , \quad \text{and} \quad \boldsymbol{d} = \begin{bmatrix} d_{11} \\ d_{22} \\ 2d_{12} \end{bmatrix} , \quad \mathcal{I} := \operatorname{span} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} ,$$

so that, in vectorial form, $\boldsymbol{\sigma}^T \boldsymbol{d} = \sigma_{ij} d_{ij}$.

The spaces S_h and D_h will be generated from bases defined on $\hat{K}$, and we will make use of the following bases on $\hat{K}$:

$$A := \operatorname{span} \begin{bmatrix} \hat{y} & 0 \\ 0 & \hat{x} \\ 0 & 0 \end{bmatrix} , B := \operatorname{span} \begin{bmatrix} \hat{x} & 0 \\ 0 & \hat{y} \\ 0 & 0 \end{bmatrix} , C := \operatorname{span} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \hat{x} & \hat{y} \end{bmatrix} . \tag{3.1}$$

Of special interest will be the following choices (S_h^i, D_h^i), $1 \le i \le 5$:

TABLE 3.1
Different cases for the discrete spaces

Case	I	II	III	IV	V
$S_\square$	$\mathcal{I}+A$	$\mathcal{I}+A$	$\mathcal{I}+C$	$\mathcal{I}+A+C$	$\mathcal{I}+A+C$
$D_\square$	$\mathcal{I}+A$	$\mathcal{I}+A+B$	$\mathcal{I}+C$	$\mathcal{I}+A+C$	$\mathcal{I}+A+B+C$
	$S_h^1 = D_h^1$	$S_h^2 \subset D_h^2$	$S_h^3 = D_h^3$	$S_h^4 = D_h^4$	$S_h^5 \subset D_h^5$

Case II corresponds to the method of mixed enhanced strains [17, 18] while Case V corresponds to the method of enhanced assumed strains [22]. We remark that $\operatorname{tr}(\mathcal{I}+A) = \operatorname{tr}(\mathcal{I}+A+C) = P_1(\hat{K})$ and $\operatorname{tr}(\mathcal{I}+C) = P_0(\hat{K})$.

In the following, all constants are generic, do not depend on the mesh size, and do not degenerate in the limit case $\lambda \longrightarrow \infty$. We restrict our analysis to the case of meshes for which F_K is an affine mapping for each element $K \in \mathcal{T}_h$. However, numerical results will also be given for the more general case of meshes of arbitrary quadrilaterals in Section 7.

From (2.7), the discrete modified Hu-Washizu formulation is as follows: find $(\boldsymbol{u}_h^\alpha, \boldsymbol{d}_h^\alpha, \boldsymbol{\sigma}_h^\alpha) \in V_h \times D_h \times S_h$ such that

$$\begin{array}{llll} a_\alpha((\boldsymbol{u}_h^\alpha, \boldsymbol{d}_h^\alpha), (\boldsymbol{v}_h, \boldsymbol{e}_h)) + b_\alpha((\boldsymbol{v}_h, \boldsymbol{e}_h), \boldsymbol{\sigma}_h^\alpha) & = & \ell(\boldsymbol{v}_h), & (\boldsymbol{v}_h, \boldsymbol{e}_h) \in V_h \times D_h , \\ b_\alpha((\boldsymbol{u}_h^\alpha, \boldsymbol{d}_h^\alpha), \boldsymbol{\tau}_h) - \frac{(1-\alpha)\gamma^2}{\lambda} c(\boldsymbol{\sigma}_h^\alpha, \boldsymbol{\tau}_h) & = & 0, & \boldsymbol{\tau}_h \in S_h \ . \end{array} \tag{3.2}$$

In contrast to the continuous setting (2.7), the discrete solution can depend on α. However for simplicity of notation, we suppress from now on the additional index α in the solution and replace $(\boldsymbol{u}_h^\alpha, \boldsymbol{d}_h^\alpha, \boldsymbol{\sigma}_h^\alpha)$ by $(\boldsymbol{u}_h, \boldsymbol{d}_h, \boldsymbol{\sigma}_h)$.

ASSUMPTION 3.1.

3.1(i) $S_h \subset D_h$

3.1(ii) $\operatorname{tr} D_h \mathbf{1} \subset D_h$

LEMMA 3.2. *Under the Assumptions 3.1(i) and 3.1(ii), the solution of (3.2) does not depend on* $\alpha \neq -\frac{\mu}{\lambda}$.

Proof. Let $(\boldsymbol{u}_h, \boldsymbol{d}_h, \boldsymbol{\sigma}_h) \in V_h \times D_h \times S_h$ be the solution of (3.2). Using (2.8) and Assumptions 3.1(i) and 3.1(ii), we find that $\mathcal{B}_\alpha \boldsymbol{e}_h$, $\mathcal{C}\boldsymbol{d}_h$, $\boldsymbol{\sigma}_h$ all belong to D_h and thus $\boldsymbol{\sigma}_h = \mathcal{C}\boldsymbol{d}_h$. The rest of the proof follows the same lines as in the continuous setting. □

REMARK 3.3. *For Lemma 3.2 the Assumption 3.1(ii) is essential, as this implies that* $\mathcal{C}D_h = D_h$. *In particular, for the choice* $S_h = D_h$ *but* $\operatorname{tr} D_h \mathbf{1} \not\subset D_h$, *the Hu-Washizu formulation with* $\alpha = 1$ *can yield bad numerical results in the limit case whereas the formulation with, for example,* $\alpha = 0$ *gives good results (see also Section 7). We note that Cases II, III and V in Table 3.1 satisfy both Assumptions 3.1(i) and 3.1(ii), but that the Cases I and IV do not satisfy 3.1(ii). Furthermore, Case IV in combination with the classical Hu-Washizu formulation* ($\alpha = 1$) *leads to the standard* Q_1*-approach.*

We note that there exists a strong link between the modified Hu-Washizu formulation (3.2), the Hellinger-Reissner, Mixed Enhanced Strain, Enhanced Assumed Strain and the classical Q_1-P_0 formulation. A detailed discussion on these equivalences can be found in [10].

4. Analysis of the modified Hu-Washizu formulation. In this section we turn to the task of establishing conditions under which the discrete saddle point problem (3.2) is uniformly stable in the limit case. For the inf-sup condition, we need some preliminary results. First we note that $\operatorname{dev} S_h$ and $\operatorname{sph} S_h$ are in general not subspaces of S_h. This is remedied by introducing the discrete deviatoric operator $\operatorname{dev_h}$ defined by $\operatorname{dev_h} S_h := P_{S_h} \operatorname{dev} S_h$, where P_{S_h} is the orthogonal projection onto S_h, and by decomposing S_h according to

$$S_h = \operatorname{dev_h} S_h \oplus \operatorname{sph_h} S_h \,. \tag{4.1}$$

Here the discrete spherical operator $\operatorname{sph_h}$ is defined in such a way that $\operatorname{sph_h} S_h$ is the orthogonal complement of $\operatorname{dev_h} S_h$. Next, we define $\widetilde{M}_h := \operatorname{tr} \operatorname{sph_h} S_h$, and remark that $\dim M_h \geq \dim \widetilde{M}_h$, where $M_h = \operatorname{tr} S_h$.

LEMMA 4.1. *The orthogonal complement* $\operatorname{sph_h} S_h$ *can be written as*

$$\operatorname{sph_h} S_h = \{\boldsymbol{\tau}_h \in S_h \mid \boldsymbol{\tau}_h = \tfrac{1}{2}(\operatorname{tr} \boldsymbol{\tau}_h)\mathbf{1}\} = \widetilde{M}_h \mathbf{1} \ .$$

Proof. We start by establishing that the spaces $\operatorname{sph_h} S_h$ and $\widetilde{M}_h \mathbf{1}$ are equal. Using the decomposition of $\operatorname{sph_h} \boldsymbol{\tau}_h$ into its deviatoric and spherical parts and the definition of $\operatorname{dev_h} S_h$, we see that

$$0 = (\operatorname{sph_h} \boldsymbol{\tau}_h, P_{S_h} \operatorname{dev} \operatorname{sph_h} \boldsymbol{\tau}_h)_0 = (\operatorname{sph_h} \boldsymbol{\tau}_h, \operatorname{dev} \operatorname{sph_h} \boldsymbol{\tau}_h)_0 = \| \operatorname{dev} \operatorname{sph_h} \boldsymbol{\tau}_h \|_0^2 \ ,$$

and thus $\operatorname{sph_h} \boldsymbol{\tau}_h = \frac{1}{2} \operatorname{tr} \operatorname{sph_h} \boldsymbol{\tau}_h \mathbf{1} \in \widetilde{M}_h \mathbf{1}$. Conversely, for $\widetilde{\boldsymbol{\tau}}_h \in \widetilde{M}_h \mathbf{1}$, there exists $\boldsymbol{\sigma}_h \in S_h$ such that

$$\widetilde{\boldsymbol{\tau}}_h = \operatorname{tr}(\operatorname{sph_h} \boldsymbol{\sigma}_h)\mathbf{1} = 2 \operatorname{sph_h} \boldsymbol{\sigma}_h \in \operatorname{sph_h} S_h \ .$$

Now, it is trivial to see that

$$\widetilde{M}_h \mathbf{1} = \widetilde{M}_h \mathbf{1} \cap S_h \subset \{\boldsymbol{\tau}_h \in S_h,\ \boldsymbol{\tau}_h = \tfrac{1}{2}\operatorname{tr}\boldsymbol{\tau}_h \mathbf{1}\}\ .$$

Using the definition of $\operatorname{dev_h} S_h$, we can write any element $\hat{\boldsymbol{\tau}}_h \in \operatorname{dev_h} S_h$ as $\hat{\boldsymbol{\tau}}_h = P_{S_h} \operatorname{dev} \boldsymbol{\sigma}_h$, $\boldsymbol{\sigma}_h \in S_h$. Let $\boldsymbol{\tau}_h = \frac{1}{2}\operatorname{tr}\boldsymbol{\tau}_h \mathbf{1} \in S_h$; then we find that

$$(\boldsymbol{\tau}_h, \hat{\boldsymbol{\tau}}_h)_0 = (\boldsymbol{\tau}_h, P_{S_h} \operatorname{dev}\boldsymbol{\sigma}_h)_0 = (\tfrac{1}{2}\operatorname{tr}\boldsymbol{\tau}_h \mathbf{1}, \operatorname{dev}\boldsymbol{\sigma}_h)_0 = 0, \quad \hat{\boldsymbol{\tau}}_h \in \operatorname{dev_h} S_h\ ,$$

yielding $\{\boldsymbol{\tau}_h \in S_h \mid \boldsymbol{\tau}_h = \frac{1}{2}\operatorname{tr}\boldsymbol{\tau}_h \mathbf{1}\} \subset \operatorname{sph_h} S_h$. □

As a result of Lemma 4.1, we find that not only the elements $\boldsymbol{\tau}_h \in \operatorname{sph_h} S_h$ and $\hat{\boldsymbol{\tau}}_h \in \operatorname{dev_h} S_h$ are orthogonal with respect to the L^2-inner product, but also their traces: that is,

$$2(\operatorname{tr}\,\boldsymbol{\tau}_h, \operatorname{tr}\,\hat{\boldsymbol{\tau}}_h)_0 = (\operatorname{tr}\,\boldsymbol{\tau}_h \mathbf{1}, \operatorname{tr}\hat{\boldsymbol{\tau}}_h \mathbf{1})_0 = 2(\boldsymbol{\tau}_h, \hat{\boldsymbol{\tau}}_h)_0 = 0\ . \tag{4.2}$$

To get a better feeling for the discrete spherical and deviatoric parts, we consider Case I in more detail. We have

$$\operatorname{dev}(\mathcal{I}+A) = \operatorname{span}\begin{bmatrix} 1 & 0 & \hat{x} & \hat{y} \\ -1 & 0 & -\hat{x} & -\hat{y} \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \operatorname{sph}(\mathcal{I}+A) = \operatorname{span}\begin{bmatrix} 1 & \hat{x} & \hat{y} \\ 1 & \hat{x} & \hat{y} \\ 0 & 0 & 0 \end{bmatrix},$$

and we find that $\dim(\mathcal{I}+A) = 5$, $\dim\operatorname{dev}(\mathcal{I}+A) = 4$, and $\dim\operatorname{sph}(\mathcal{I}+A) = 3$. We point out that in general $\operatorname{sph_h} S_h \neq P_{S_h} \operatorname{sph} S_h$; for example

$$\operatorname{dev_h}(\mathcal{I}+A) = \operatorname{span}\begin{bmatrix} 1 & 0 & \hat{y} & 0 \\ -1 & 0 & 0 & \hat{x} \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad P_{S_h}\operatorname{sph}(\mathcal{I}+A) = \operatorname{span}\begin{bmatrix} 1 & \hat{y} & 0 \\ 1 & 0 & \hat{x} \\ 0 & 0 & 0 \end{bmatrix},$$

whereas $\operatorname{sph_h}(\mathcal{I}+A) = \operatorname{span}[1,1,0]^T$. In the continuous setting, the definition of the deviatoric part yields that the trace of the deviatoric part is equal to zero. This is not the case for an element in $\operatorname{dev_h} S_h$.

For $\boldsymbol{\tau} \in S$, the norm of the trace is bounded by $\|\operatorname{tr}\boldsymbol{\tau}\|_0^2 \leq 2\|\boldsymbol{\tau}\|_0^2$. The following lemma shows that a stronger bound holds for the norm of the trace of an element in $\operatorname{dev_h} S_h$.

Lemma 4.2. *There exists a* $\omega < 1$ *such that*

$$\|\operatorname{tr}\boldsymbol{\tau}_h\|_0^2 \leq 2\omega\|\boldsymbol{\tau}_h\|_0^2, \quad \boldsymbol{\tau}_h \in \operatorname{dev_h} S_h\ ,$$

and thus $(\mathcal{C}^{-1}\boldsymbol{\tau}_h, \boldsymbol{\tau}_h)_0 \geq \frac{1-\omega}{2\mu}\|\boldsymbol{\tau}_h\|_0^2$, $\boldsymbol{\tau}_h \in \operatorname{dev_h} S_h$.

Proof. The proof is based on a discrete norm equivalence on the reference element. To start, we define $\|\boldsymbol{\tau}\|_{*;\hat{K}} := \|\operatorname{dev}\boldsymbol{\tau}\|_{0;\hat{K}}$. It is trivial to see that $\|\cdot\|_*$ defines a seminorm on $(L^2(\hat{K}))^{2\times 2}$ and that it is zero if and only if $\operatorname{dev}\boldsymbol{\tau}$ is zero. Given $\boldsymbol{\tau}_h \in \operatorname{dev_h} S_\square$ and $\operatorname{dev}\boldsymbol{\tau}_h = 0$, we find by means of (4.1) and Lemma 4.1 that $\boldsymbol{\tau}_h \in \operatorname{dev_h} S_\square \cap \operatorname{sph_h} S_\square = \{0\}$. Thus the seminorm $\|\cdot\|_{*;K}$ restricted to $\operatorname{dev_h} S_\square$ is a norm. The dimension of the space $\operatorname{dev_h} S_\square$ is finite and thus $\|\cdot\|_{*;\hat{K}}$ is equivalent to $\|\cdot\|_{0;\hat{K}}$. Using the definition of S_h and the special structure of the isoparametric element map F_K, the equivalence of $\|\cdot\|_*^2 := \sum_{K\in\mathcal{T}_h}\|\cdot\|_{*;K}^2$ and $\|\cdot\|_0^2$ on $\operatorname{dev_h} S_h$ is established. Now let $0 < \beta < 1$ be such that $\|\boldsymbol{\tau}_h\|_*^2 \geq \beta\|\boldsymbol{\tau}_h\|_0^2$, $\boldsymbol{\tau}_h \in \operatorname{dev_h} S_h$; then a straightforward computation reveals that

$$2(1-\beta)\|\boldsymbol{\tau}_h\|_0^2 \geq \|\operatorname{tr}\boldsymbol{\tau}_h\|_0^2, \quad (\mathcal{C}^{-1}\boldsymbol{\tau}_h, \boldsymbol{\tau}_h)_0 \geq \frac{1}{2\mu}(\|\boldsymbol{\tau}_h\|_0^2 - 2\gamma(1-\beta)\|\boldsymbol{\tau}_h\|_0^2)\ .$$

□

We can now use these preliminary results to establish a uniform inf-sup condition.

ASSUMPTION 4.3. *The following assumptions are essential for uniform stability in the discrete case:*

4.3 (i) $\|P_{S_h}\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_0 \geq c\|\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_0$,

4.3 (ii) $\max(-\frac{c_l\mu}{\lambda}, 1-\frac{c_l}{\omega}) \leq \alpha \leq \min(\frac{C_u\mu}{\lambda}, 1)$, $0 < c_l < 1$, $0 < C_u < \infty$,

4.3 (iii) $(V_h, \widetilde{M}_h)$ *forms a stable Stokes pairing,*

where ω is the bound from Lemma 4.2. In the following the generic constants depend on c_l and C_u. It is easy to verify that $\alpha = 0$ or when $\mu \leq \lambda$, $\alpha = \mu/\lambda$ are possible choices that satisfy Assumption 4.3 (ii) with some suitable c_l and C_u. As we will see later, of special interest for Cases I and IV will be a negative value of α. However, as Assumption 4.3(ii) indicates, the choice of a negative α can lead to the loss of coercivity.

Roughly speaking, Assumption 4.3 (i) says that S_h has to be large enough and Assumption 4.3 (iii), that S_h has to be small enough. We point out that Assumption 4.3 (iii) is weaker than the assumption that (V_h, M_h) forms a stable Stokes pairing. For example, in Case I we find that $\operatorname{tr}(\mathcal{I}+A) = P_1(\hat{K})$, whereas $\operatorname{tr}\operatorname{sph_h}(\mathcal{I}+A) = P_0(\hat{K})$.

The discrete kernel associated with the bilinear form $b_\alpha(\cdot,\cdot)$ is given by

$$Z_h := \{(\boldsymbol{v}_h, \boldsymbol{e}_h) \in V_h \times D_h \mid P_{S_h}\boldsymbol{\varepsilon}(\boldsymbol{v}_h) = P_{S_h}(2\mu\mathcal{C}^{-1}\boldsymbol{e}_h + \alpha\gamma \operatorname{tr}\boldsymbol{e}_h\mathbf{1})\} \ .$$

LEMMA 4.4. *Under the Assumptions 4.3 (i) and 4.3 (ii), the bilinear form $a_\alpha(\cdot,\cdot)$ is uniformly coercive on $Z_h \times Z_h$ and uniformly continuous on $(V_h\times D_h)\times(V_h\times D_h)$.*

Proof. Let $(\boldsymbol{v}_h, \boldsymbol{e}_h) \in Z_h$. Assuming 4.3 (i) and 4.3 (ii), we find that

$$\begin{aligned}\|\boldsymbol{v}_h\|_1^2 \leq C\|\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_0^2 \leq C\|P_{S_h}\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_0^2 = C\|P_{S_h}(2\mu\mathcal{C}^{-1}\boldsymbol{e}_h + \alpha\gamma(\operatorname{tr}\boldsymbol{e}_h)\mathbf{1})\|_0^2 \\ \leq C\|\boldsymbol{e}_h\|_0^2 \leq C a_\alpha((\boldsymbol{v}_h,\boldsymbol{e}_h),(\boldsymbol{v}_h,\boldsymbol{e}_h)) \ .\end{aligned}$$

□

LEMMA 4.5. *Under the Assumptions 3.1 (i), 4.3 (ii) and 4.3 (iii), the bilinear form $b_\alpha(\cdot,\cdot)$ satisfies a uniform inf-sup condition.*

Proof. The proof is based on the decomposition of $\boldsymbol{\tau}_h$ into discrete deviatoric and spherical parts, that is,

$$\boldsymbol{\tau}_h = \operatorname{dev_h}\boldsymbol{\tau}_h + \operatorname{sph_h}\boldsymbol{\tau}_h = \operatorname{dev_h}\boldsymbol{\tau}_h + q_h\mathbf{1} \ ,$$

where $\operatorname{dev_h}\boldsymbol{\tau}_h \in \operatorname{dev_h} S_h$, $\operatorname{sph_h}\boldsymbol{\tau}_h \in \operatorname{sph_h} S_h$ and $q_h \in \widetilde{M}_h$. Under the Assumption 4.3 (iii) we can find, for each $q_h \in \widetilde{M}_h$, a displacement $\boldsymbol{v}_{q_h} \in V_h$ such that

$$(\operatorname{div}\boldsymbol{v}_{q_h}, q_h)_0 = \|q_h\|_0^2, \quad \|\boldsymbol{v}_{q_h}\|_1 \leq C\|q_h\|_0 \ . \tag{4.3}$$

Using the unique decomposition of $P_{S_h}\boldsymbol{\varepsilon}(\boldsymbol{v}_{q_h})$ into its discrete spherical and deviatoric parts, so that $P_{S_h}\boldsymbol{\varepsilon}(\boldsymbol{v}_{q_h}) = \operatorname{sph_h} P_{S_h}\boldsymbol{\varepsilon}(\boldsymbol{v}_{q_h}) + \operatorname{dev_h} P_{S_h}\boldsymbol{\varepsilon}(\boldsymbol{v}_{q_h})$, we define $\boldsymbol{e}_{\boldsymbol{\tau}_h} \in D_h$ by

$$\boldsymbol{e}_{\boldsymbol{\tau}_h} := \operatorname{dev_h} P_{S_h}\boldsymbol{\varepsilon}(\boldsymbol{v}_{q_h}) - \beta\operatorname{dev_h}\boldsymbol{\tau}_h, \quad \beta > 0 \ .$$

By means of (4.3) and the fact that $\|\operatorname{dev_h} P_{S_h}\boldsymbol{\varepsilon}(\boldsymbol{v}_{q_h})\|_0 \leq \|\boldsymbol{\varepsilon}(\boldsymbol{v}_{q_h})\|_0 \leq C\|\boldsymbol{v}_{q_h}\|_1$, we can bound the norm of $(\boldsymbol{v}_{q_h}, \boldsymbol{e}_{\boldsymbol{\tau}_h})$ by

$$\|\boldsymbol{v}_{q_h}\|_1^2 + \|\boldsymbol{e}_{\boldsymbol{\tau}_h}\|_0^2 \leq C(\|q_h\|_0^2 + \beta^2\|\operatorname{dev_h}\boldsymbol{\tau}_h\|_0^2) \leq C(1+\beta^2)\|\boldsymbol{\tau}_h\|_0^2 \ .$$

To obtain the inf-sup condition, we use the equivalence (2.9) and consider the two terms of $b_\alpha(\cdot,\cdot)$ separately. We start by obtaining a lower bound for $b_1((\boldsymbol{v}_{q_h}, \boldsymbol{e}_{\boldsymbol{\tau}_h}), \boldsymbol{\tau}_h)$:

$$\begin{aligned} b_1((\boldsymbol{v}_{q_h}, \boldsymbol{e}_{\boldsymbol{\tau}_h}), \boldsymbol{\tau}_h) &= (P_{S_h}\boldsymbol{\varepsilon}(\boldsymbol{v}_{q_h}) - \boldsymbol{e}_{\boldsymbol{\tau}_h}, \boldsymbol{\tau}_h)_0 = (\mathrm{sph_h}\, P_{S_h}\boldsymbol{\varepsilon}(\boldsymbol{v}_{q_h}) + \beta\, \mathrm{dev_h}\, \boldsymbol{\tau}_h, \boldsymbol{\tau}_h)_0 \\ &= (\mathrm{sph_h}\, P_{S_h}\boldsymbol{\varepsilon}(\boldsymbol{v}_{q_h}), q_h\mathbf{1})_0 + \beta\|\, \mathrm{dev_h}\, \boldsymbol{\tau}_h\|_0^2 \\ &= (\boldsymbol{\varepsilon}(\boldsymbol{v}_{q_h}), q_h\mathbf{1})_0 + \beta\|\, \mathrm{dev_h}\, \boldsymbol{\tau}_h\|_0^2 \\ &= (\mathrm{div}\, \boldsymbol{v}_{q_h}, q_h)_0 + \beta\|\, \mathrm{dev_h}\, \boldsymbol{\tau}_h\|_0^2 = \|q_h\|_0^2 + \beta\|\, \mathrm{dev_h}\, \boldsymbol{\tau}_h\|_0^2 \ . \end{aligned}$$

Next, we bound $c(\boldsymbol{e}_{\boldsymbol{\tau}_h}, \boldsymbol{\tau}_h)$ by applying Lemma 4.2, (4.2) and (4.3):

$$\begin{aligned} c(\boldsymbol{e}_{\boldsymbol{\tau}_h}, \boldsymbol{\tau}_h) &= (\mathrm{tr}\, \boldsymbol{e}_{\boldsymbol{\tau}_h}, \mathrm{tr}\, \boldsymbol{\tau}_h)_0 = (\mathrm{tr}\,(\mathrm{dev_h}\, P_{S_h}\boldsymbol{\varepsilon}(\boldsymbol{v}_{q_h})) - \beta\, \mathrm{tr}\, \mathrm{dev_h}\, \boldsymbol{\tau}_h, \mathrm{tr}\, \mathrm{dev_h}\, \boldsymbol{\tau}_h)_0 \\ &= (\mathrm{tr}\, P_{S_h}\boldsymbol{\varepsilon}(\boldsymbol{v}_{q_h}), \mathrm{tr}\, \mathrm{dev_h}\, \boldsymbol{\tau}_h)_0 - \beta\|\, \mathrm{tr}\, \mathrm{dev_h}\, \boldsymbol{\tau}_h\|_0^2 \\ &\geq -C\|q_h\|_0\|\, \mathrm{dev_h}\, \boldsymbol{\tau}_h\|_0 - 2\beta\omega\|\, \mathrm{dev_h}\, \boldsymbol{\tau}_h\|_0^2 \ . \end{aligned}$$

Combining the lower bounds for $b_1((\boldsymbol{v}_{q_h}, \boldsymbol{e}_{\boldsymbol{\tau}_h}), \boldsymbol{\tau}_h)$ and $c(\boldsymbol{e}_{\boldsymbol{\tau}_h}, \boldsymbol{\tau}_h)$, and using $2\gamma \leq 1$ we find that

$$b_\alpha((\boldsymbol{v}_{q_h}, \boldsymbol{e}_{\boldsymbol{\tau}_h}), \boldsymbol{\tau}_h) \geq \|q_h\|_0^2 - 2C\|q_h\|_0\|\, \mathrm{dev_h}\, \boldsymbol{\tau}_h\|_0 + \beta(1 - \omega(1-\alpha))\|\, \mathrm{dev_h}\, \boldsymbol{\tau}_h\|_0^2 \ .$$

Assumption 4.3 (ii) gives $(1 - \omega(1-\alpha)) \geq 1 - c_l > 0$, and we can apply Young's inequality to find

$$b_\alpha((\boldsymbol{v}_{q_h}, \boldsymbol{e}_{\boldsymbol{\tau}_h}), \boldsymbol{\tau}_h) \geq (1 - C\epsilon)\|q_h\|_0^2 + (\beta(1-c_l) - C\epsilon^{-1})\|\, \mathrm{dev_h}\, \boldsymbol{\tau}_h\|_0^2 \ .$$

For $0 < \epsilon$ small enough and $\beta < \infty$ large enough, we obtain $b_\alpha((\boldsymbol{v}_{q_h}, \boldsymbol{e}_{\boldsymbol{\tau}_h}), \boldsymbol{\tau}_h) \geq c(\|q_h\|_0^2 + \|\, \mathrm{dev_h}\, \boldsymbol{\tau}_h\|_0^2) \geq c\|\boldsymbol{\tau}_h\|_0^2$. □

We now formulate our main result. The following theorem provides optimal a priori estimates for the displacement, strain, and stress.

THEOREM 4.6. *Under the Assumptions 3.1 (i), 4.3 (i)–4.3 (iii), the discretization error* $\eta_h^2 := \|\boldsymbol{u} - \boldsymbol{u}_h\|_1^2 + \|\boldsymbol{d} - \boldsymbol{d}_h\|_0^2 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0^2$*, where* $(\boldsymbol{u}_h, \boldsymbol{d}_h, \boldsymbol{\sigma}_h)$ *is the solution of (3.2), is bounded by the best approximation error*

$$\eta_h^2 \leq C\left(\inf_{\boldsymbol{v}_h \in V_h} \|\boldsymbol{u} - \boldsymbol{v}_h\|_1^2 + \inf_{\boldsymbol{e}_h \in D_h} \|\boldsymbol{d} - \boldsymbol{e}_h\|_0^2 + \inf_{\boldsymbol{\tau}_h \in S_h} \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_0^2\right) \ .$$

Proof. The a priori estimate results from the continuity of the bilinear forms and Lemmas 2.2, 4.4 and 4.5; see, for example, [8]. In particular, it is seen that $\ker B_h^T = \{\mathbf{0}\}$, as in the continuous case, and $\|c(\cdot,\cdot)\|$ is bounded, independently of λ. These conditions suffice to establish the uniqueness of $\boldsymbol{\sigma}_h$ and the uniform error estimate (see [8], Section II and [4]). □

In the rest of this section, we consider the Assumptions 4.3 (i) and 4.3 (ii) in more detail. To verify Assumption 4.3 (i) we consider the reference element and observe that

$$\boldsymbol{\varepsilon}(V_\square) := \mathrm{span}\begin{bmatrix} 1 & 0 & 0 & \hat{y} & 0 \\ 0 & 1 & 0 & 0 & \hat{x} \\ 0 & 0 & 1 & \hat{x} & \hat{y} \end{bmatrix} \ ,$$

where $V_\square$ is the space spanned by the standard bilinear polynomials on the reference element. A straightforward computation shows that on the reference element and thus on all elements $K \in \mathcal{T}_h$, Assumption 4.3 (i) is satisfied for all choices of $S_\square$ in Table 3.1.

We recall that Assumption 4.3 (iii) is on $\widetilde{M}_h$, which must be such that it forms with V_h a stable Stokes pair. Now for all our five choices of pairings (D_h^i, S_h^i), $1 \le i \le 5$, we find that $\widetilde{M}_h^i$ is given by

$$\widetilde{M}_h^i = \{q \in L_0^2(\Omega) \mid \ q|_K \in P_0(K),\ K \in \mathcal{T}_h\}, \quad 1 \le i \le 5\ .$$

Using bilinear elements for the displacements, it is well known that the pairing $(V_h, \widetilde{M}_h^i)$ does not satisfy Assumption 4.3 (iii), and that checkerboard modes might be observed (see, for example, [15]). Thus it is necessary that $\widetilde{M}_h$ be a proper subset of $\widetilde{M}_h^i$. There are different ways in which to overcome this difficulty. One option is to work with macro-elements and to extract from $\widetilde{M}_h^i$ the checkerboard mode on each macro-element, as in [15] (Section II.3). The restrictions of functions in $\widetilde{M}_h^i$ to a macro-element are spanned by the four functions depicted in Figure 4.1. The functions having the signs indicated in Figure 4.1 (d) are the local checkerboard modes. To obtain a stable pairing, we have to modify the space $\mathrm{sph_h}\, S_h^i$ but not $\mathrm{dev_h}\, S_h^i$.

A reduction in the dimension of $\widetilde{M}_h^i$ results in a smaller dimension of S_h^i, and thus for our choices Assumption 4.3 (i) cannot be verified locally on each element, but must be done on macro-elements. Using the fact that V_h comprises continuous displacements, and taking into account the rigid body motions, we find that the dimension of $\boldsymbol{\varepsilon}(V_h)$ restricted to one macro-element is 15 and not 4×5. The modification of $\widetilde{M}_h^i$ reduces the dimension of S_h^i on each macro-element by one, and thus for all our examples the dimension of S_h^i restricted to one macro-element is not less than 19. Moreover, a straightforward computation on the macro-element shows that for all S_h^i, $1 \le i \le 5$, Assumption 4.3 (i) is satisfied even if we work on macro-elements and reduce the dimension of S_h^i restricted to a macro-element by one.

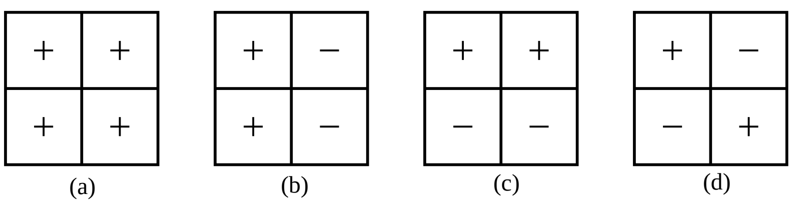

Fig. 4.1. *Restrictions of the basis functions of $\widetilde{M}_h^i$ to a macro-element with $\pm$ indicating the sign inside the elements*

5. A priori estimate for the displacement. Due to the local definition of S_h and D_h, the discrete stress and the discrete strain variables can be locally eliminated in our modified Hu-Washizu formulation. Static condensation in (2.7) yields a displacement based formulation of the following form: find $\boldsymbol{u}_h \in V_h$ such that

$$(Q_h \boldsymbol{\varepsilon}(\boldsymbol{v}_h), \mathcal{C}_h Q_h \boldsymbol{\varepsilon}(\boldsymbol{u}_h))_0 = \ell(\boldsymbol{v}_h), \quad \boldsymbol{v}_h \in V_h\ , \tag{5.1}$$

where Q_h is a suitable local projection and $\mathcal{C}_h$ is some positive-definite symmetric operator. The symmetry results from the symmetry in (2.7). We note that the operators as well as the displacement depend on S_h, D_h and α. For simplicity of notation, we do not indicate this dependence.

In the following, we provide in a first step some abstract conditions under which (5.1) gives optimal λ-independent a priori estimates. In a second step, we verify these conditions for all our five examples. The optimal a priori estimate is based on some preliminary results which can be found in the literature. For convenience of the reader, we present some of these results in a form which is suitable for our analysis. We assume that the triangulation $\mathcal{T}_h$ has a macro-element structure and that there is a local checkerboard free subspace $\widetilde{M}_h^s \subset \widetilde{M}_h$. The space $\widetilde{M}_h^s$ is the orthogonal complement of $\widetilde{M}_h^u$, which is spanned by the local checkerboard modes (Figure 4.1 (d)). Moreover, we assume that the pairing $(V_h^s, \widetilde{M}_h^s)$ satisfies a uniform inf-sup condition, where

$$V_h^s := \{\boldsymbol{v}_h \in V_h \mid (\operatorname{div} \boldsymbol{v}_h, q_h)_0 = 0,\ q_h \in \widetilde{M}_h^u\} \ ,$$

and that both subspaces $\widetilde{M}_h^s$ and V_h^s satisfy suitable approximation properties [15, 6]. In the following, we assume that $\boldsymbol{u}$ is the solution of (2.1)–(2.4), $\boldsymbol{u}_h$ is the first component of the solution of (3.2), and we set $W := [H^2(\Omega) \cap H_0^1(\Omega)]^2$. We recall that Fortin's interpolation operator $\boldsymbol{I}_h^F : W \longrightarrow V_h^s$ for the stable Stokes pairing $(V_h^s, \widetilde{M}_h^s)$ is then given by the problem of finding $(\boldsymbol{I}_h^F \boldsymbol{w}, p_h) \in V_h^s \times \widetilde{M}_h^s$ such that

$$\begin{aligned} (\nabla \boldsymbol{I}_h^F \boldsymbol{w}, \nabla \boldsymbol{z}_h)_0 + (\operatorname{div} \boldsymbol{z}_h, p_h)_0 &= (\nabla \boldsymbol{w}, \nabla \boldsymbol{z}_h)_0, \qquad \boldsymbol{z}_h \in V_h^s \ , \\ (\operatorname{div} \boldsymbol{I}_h^F \boldsymbol{w}, q_h)_0 &= (\operatorname{div} \boldsymbol{w}, q_h)_0, \qquad q_h \in \widetilde{M}_h^s \ . \end{aligned} \tag{5.2}$$

LEMMA 5.1. *Under the regularity assumption (2.6), the operator I_h^F satisfies the approximation property*

$$\|\boldsymbol{u} - \boldsymbol{I}_h^F \boldsymbol{u}\|_1 + \lambda \|\operatorname{div} \boldsymbol{u} - \Pi_h(\operatorname{div} \boldsymbol{I}_h^F \boldsymbol{u})\|_0 \le Ch\|f\|_0 \ ,$$

where Π_h is the L^2-projection on $\widetilde{M}_h$.

Proof. Using the stability of the saddlepoint problem (5.2) and (2.6), the upper bound for $\|\boldsymbol{u} - \boldsymbol{I}_h^F \boldsymbol{u}\|_1$ is standard. There exists a $\tilde{\boldsymbol{u}} \in W$ such that $\operatorname{div} \boldsymbol{u} = \operatorname{div} \tilde{\boldsymbol{u}}$ and $\|\tilde{\boldsymbol{u}}\|_2 \le C\|\operatorname{div} \boldsymbol{u}\|_1$, see, for example, [2]. Observing the definitions of V_h^s and of I_h^F, we find that

$$(\operatorname{div} \boldsymbol{I}_h^F \boldsymbol{u}, q_h)_0 = (\operatorname{div} \boldsymbol{I}_h^F \tilde{\boldsymbol{u}}, q_h)_0, \quad q_h \in \widetilde{M}_h \ ,$$

and thus $\Pi_h(\operatorname{div} \boldsymbol{I}_h^F \boldsymbol{u}) = \Pi_h(\operatorname{div} \boldsymbol{I}_h^F \tilde{\boldsymbol{u}})$. In terms of (2.6), we obtain

$$\begin{aligned} \|\operatorname{div} \boldsymbol{u} - \Pi_h(\operatorname{div} \boldsymbol{I}_h^F \boldsymbol{u})\|_0 &= \|\operatorname{div} \tilde{\boldsymbol{u}} - \Pi_h(\operatorname{div} \boldsymbol{I}_h^F \tilde{\boldsymbol{u}})\|_0 \\ &\le \|\operatorname{div}(\tilde{\boldsymbol{u}} - \boldsymbol{I}_h^F \tilde{\boldsymbol{u}})\|_0 + \|\operatorname{div} \tilde{\boldsymbol{u}} - \Pi_h \operatorname{div} \tilde{\boldsymbol{u}}\|_0 \\ &\le C(\|\tilde{\boldsymbol{u}} - \boldsymbol{I}_h^F \tilde{\boldsymbol{u}}\|_1 + h\|\tilde{\boldsymbol{u}}\|_2) \le Ch\|\operatorname{div} \boldsymbol{u}\|_1 \le \frac{C}{\lambda} h\|f\|_0 \ . \end{aligned}$$

□

In this section, we provide some assumptions on Q_h and $\mathcal{C}_h$ under which optimal a priori estimates for (5.1) can be established. These assumptions can be easily verified for all given examples.

ASSUMPTION 5.2. *The following assumptions are essential for λ-independent constants in the a priori estimates: for $\boldsymbol{v}_h \in V_h$,*

5.2 (i) $\|\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_0^2 \le C(Q_h \boldsymbol{\varepsilon}(\boldsymbol{v}_h), \mathcal{C}_h Q_h \boldsymbol{\varepsilon}(\boldsymbol{v}_h))_0$*;*

5.2 (ii) $(Q_h \boldsymbol{\varepsilon}(\boldsymbol{v}_h), \mathcal{C}_h Q_h \boldsymbol{\varepsilon}(\boldsymbol{w}_h))_0 = (\boldsymbol{\varepsilon}(\boldsymbol{v}_h), \mathcal{C}_h Q_h \boldsymbol{\varepsilon}(\boldsymbol{w}_h))_0$*,* $\boldsymbol{w}_h \in V_h$*;*

5.2 (iii) $\|(\mathcal{C}_h Q_h - \boldsymbol{\Pi}_h \mathcal{C})\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_0 \le C\|(\mathbf{Id} - \boldsymbol{\Pi}_\mathrm{h})\boldsymbol{\varepsilon}(\boldsymbol{v}_\mathrm{h})\|_0$*, where* $\boldsymbol{\Pi}_h$ *is the component-wise L^2-projector.*

Theorem 5.3. *Under the Assumptions 5.2(i)-(iii) and (2.6), the upper bound*

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_1 \le Ch\|\boldsymbol{f}\|_0$$

for the discretization error holds, where $C<\infty$ is independent of λ and h.

Proof. In a first step, we bound $\|\boldsymbol{v}_h-\boldsymbol{u}_h\|_1$, $\boldsymbol{v}_h \in V_h$, in terms of $\|\boldsymbol{v}_h-\boldsymbol{u}\|_1$. Using Korn's inequality and Assumptions 5.2(i)–(ii) and (2.5), we find that

$$\begin{aligned}\|\boldsymbol{v}_h-\boldsymbol{u}_h\|_1^2 &\le C\|\boldsymbol{\varepsilon}(\boldsymbol{v}_h-\boldsymbol{u}_h)\|_0^2 \le C(Q_h\boldsymbol{\varepsilon}(\boldsymbol{v}_h-\boldsymbol{u}_h),\mathcal{C}_hQ_h\boldsymbol{\varepsilon}(\boldsymbol{v}_h-\boldsymbol{u}_h))_0\\ &= C\left((\boldsymbol{\varepsilon}(\boldsymbol{v}_h-\boldsymbol{u}_h),\mathcal{C}_hQ_h\boldsymbol{\varepsilon}(\boldsymbol{v}_h))_0-(\boldsymbol{\varepsilon}(\boldsymbol{v}_h-\boldsymbol{u}_h),\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}))_0\right)\\ &\le C\|\boldsymbol{v}_h-\boldsymbol{u}_h\|_1\|\mathcal{C}_hQ_h\boldsymbol{\varepsilon}(\boldsymbol{v}_h)-\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u})\|_0\ .\end{aligned}$$

In terms of Assumption 5.2(iii), the triangle inequality, the properties of I_h^F, and Lemma 5.1, we obtain

$$\begin{aligned}\|\boldsymbol{u}-\boldsymbol{u}_h\|_1 &\le C(\|(\mathcal{C}_hQ_h-\boldsymbol{\Pi}_h\mathcal{C})\boldsymbol{\varepsilon}(\boldsymbol{I}_h^F\boldsymbol{u})\|_0+\|\mathcal{C}(\boldsymbol{\Pi}_h\boldsymbol{\varepsilon}(\boldsymbol{I}_h^F\boldsymbol{u})-\boldsymbol{\varepsilon}(\boldsymbol{u}))\|_0+\|\boldsymbol{u}-\boldsymbol{I}_h^F\boldsymbol{u}\|_1)\\ &\le C(\lambda\|\Pi_h(\operatorname{div}\boldsymbol{I}_h^F\boldsymbol{u})-\operatorname{div}\boldsymbol{u}\|_0+\|\boldsymbol{\varepsilon}(\boldsymbol{u})-\boldsymbol{\Pi}_h\boldsymbol{\varepsilon}(\boldsymbol{u})\|_0+\|\boldsymbol{u}-\boldsymbol{I}_h^F\boldsymbol{u}\|_1)\le Ch\|f\|_0\ .\end{aligned}$$

□

Next, we consider the operators $\mathcal{C}_h$ and Q_h in more detail. We start with an orthogonal decomposition of S_h and define $S_h^c := \{\boldsymbol{\tau}\in S_h \mid \mathcal{C}\boldsymbol{\tau}\in S_h\}$, and

$$S_h = S_h^c \oplus S_h^t\ .$$

We note that $S_h^t \subset \mathrm{dev_h}\, S_h$, $\mathrm{sph_h}\, S_h \subset S_h^c$ and $\mathrm{tr}(\mathrm{sph_h}\, S_h) = \mathrm{tr}\, S_h^c$.

Assumption 5.4. $(\boldsymbol{\tau}_h^t,\boldsymbol{\omega}_h^t)_0 = (\mathrm{tr}\,\boldsymbol{\tau}_h^t,\mathrm{tr}\,\boldsymbol{\omega}_h^t)_0$, $\boldsymbol{\tau}_h^t,\boldsymbol{\omega}_h^t\in S_h^t$

It is straightforward to verify this assumption for all our Cases I–V. For Case III, we find $S_h^t=\{\boldsymbol{0}\}$, and for all other cases, S_h^t is spanned locally by the basis A in (3.1). Let $\hat{\boldsymbol{\tau}}_h,\hat{\boldsymbol{\omega}}_h\in S_\square^t$ be given by $\hat{\tau}_{h_{11}}=a\hat{y}$, $\hat{\tau}_{h_{22}}=b\hat{x}$, $\hat{\tau}_{h_{12}}=\hat{\tau}_{h_{21}}=0$, $\hat{\omega}_{h_{11}}=c\hat{y}$, $\hat{\omega}_{h_{22}}=d\hat{x}$, $\hat{\omega}_{h_{12}}=\hat{\omega}_{h_{21}}=0$, then we find $\mathrm{tr}\,\hat{\boldsymbol{\tau}}_h=a\hat{y}+b\hat{x}$ and $\mathrm{tr}\,\hat{\boldsymbol{\omega}}_h=c\hat{y}+d\hat{x}$. Using the definition of the reference element we find that

$$(\hat{\boldsymbol{\tau}}_h,\hat{\boldsymbol{\omega}}_h)_{0;\hat{K}} = \frac{4}{3}(ac+bd) = (\mathrm{tr}\,\hat{\boldsymbol{\tau}}_h,\mathrm{tr}\,\hat{\boldsymbol{\omega}}_h)_{0;\hat{K}}\ ,$$

and thus Assumption 5.4 is satisfied for all our cases.

Lemma 5.5. *For $\alpha\neq-\frac{\mu}{\lambda}$ the operator $\mathcal{C}_h$ is given by:*

(a) under the Assumptions 3.1(i)–(ii) and 5.4

$$\mathcal{C}_h\boldsymbol{e}_h = \mathcal{C}P_{S_h^c}\boldsymbol{e}_h + \frac{2\mu}{1-\gamma}P_{S_h^t}\boldsymbol{e}_h\ .$$

(b) under the assumptions $S_h=D_h$, $\alpha\neq-2\mu/\lambda$ and 5.4,

$$\mathcal{C}_h\boldsymbol{e}_h = \mathcal{C}P_{S_h^c}\boldsymbol{e}_h + \frac{4(\mu+\lambda)^2(2\mu+\alpha\lambda)}{\lambda^2+(2\mu+3\lambda)(2\mu+\alpha\lambda)}P_{S_h^t}\boldsymbol{e}_h\ ,$$

where $P_{S_h^c}$ and $P_{S_h^t}$ are orthogonal projections onto the spaces S_h^c and S_h^t, respectively.

The operator Q_h is given by $Q_h\boldsymbol{\varepsilon}(\boldsymbol{v}_h)=P_{S_h}\boldsymbol{\varepsilon}(\boldsymbol{v}_h)$ for both cases (a) and (b). Then the displacement based formulation (5.1) is equivalent to the three field formulation (3.2).

Proof. The main idea of the proof is to use the orthogonal decomposition of S_h as given above and to apply static condensation. In the case that Assumptions 3.1(i)–(ii) hold, Lemma 3.2 yields $\boldsymbol{\sigma}_h = \mathcal{C}\boldsymbol{d}_h$. In terms of the decomposition given above, we can write $\boldsymbol{\sigma}_h = \boldsymbol{\sigma}_h^c + \boldsymbol{\sigma}_h^t$, $\boldsymbol{\sigma}_h^c \in S_h^c$, $\boldsymbol{\sigma}_h^t \in S_h^t$. Using $\boldsymbol{d}_h = \mathcal{C}^{-1}\boldsymbol{\sigma}_h$, Assumption 5.4, and the orthogonal decomposition of S_h in the second equation of (3.2), we get

$$\boldsymbol{\sigma}_h^c = \mathcal{C}P_{S_h^c}\boldsymbol{\varepsilon}(\boldsymbol{u}_h), \text{ and } \boldsymbol{\sigma}_h^t = \frac{2\mu}{1-\gamma}P_{S_h^t}\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\ . \tag{5.3}$$

We point out that $0 < 2\gamma \leq 1$ independently of λ. If $S_h = D_h$, the first equation in (3.2) yields $P_{S_h}(\mathcal{B}_\alpha(\mathcal{C}\boldsymbol{d}_h - \boldsymbol{\sigma}_h)) = 0$. Using the decomposition for $\boldsymbol{d}_h$, we find that $\boldsymbol{d}_h = \boldsymbol{d}_h^c + \boldsymbol{d}_h^t$, $\boldsymbol{d}_h^c \in S_h^c$, $\boldsymbol{d}_h^t \in S_h^t$, and

$$\boldsymbol{d}_h^c = \mathcal{C}^{-1}\boldsymbol{\sigma}_h^c\,, \quad P_{S_h^t}(\mathcal{B}_\alpha(\mathcal{C}\boldsymbol{d}_h^t - \boldsymbol{\sigma}_h^t)) = \mathbf{0}\ . \tag{5.4}$$

Assumption 5.4 yields

$$\boldsymbol{d}_h^t = \frac{1+\gamma(\alpha-1)}{\alpha\lambda+2\mu}\boldsymbol{\sigma}_h^t\ . \tag{5.5}$$

In terms of (5.4) and (5.5), we can eliminate $\boldsymbol{d}_h^c$ and $\boldsymbol{d}_h^t$ from the second equation of (3.2). The elimination of $\boldsymbol{d}_h^c$ gives $\boldsymbol{\sigma}_h^c = \mathcal{C}P_{S_h^c}\boldsymbol{\varepsilon}(\boldsymbol{u}_h)$, and by means of (5.5), we obtain

$$\boldsymbol{\sigma}_h^t = \frac{4(\mu+\lambda)^2(2\mu+\alpha\lambda)}{\lambda^2+(2\mu+3\lambda)(2\mu+\alpha\lambda)}P_{S_h^t}\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\,. \tag{5.6}$$

□

Remark 5.6. *The theoretical analysis requires that we work with the discrete space $S_h \subset S_0$. Using the bigger space $\hat{S}_h := S_h \oplus span\{\mathbf{1}\}$, $dim\hat{S}_h = dim S_h + 1$, we find that $(P_{\hat{S}_h}\boldsymbol{\varepsilon}(\boldsymbol{v}_h), \mathbf{1})_0 = (\boldsymbol{\varepsilon}(\boldsymbol{v}_h), \mathbf{1})_0 = (\operatorname{div}\boldsymbol{v}_h, 1)_0 = 0$, $\boldsymbol{v}_h \in V_h$. Thus Q_h can be locally computed on each element $K \in \mathcal{T}_h$ by*

$$Q_h\boldsymbol{\varepsilon}(\boldsymbol{v}_h) = P_{\hat{S}_h}\boldsymbol{\varepsilon}(\boldsymbol{v}_h) \in S_h, \quad \boldsymbol{v}_h \in V_h\ .$$

Remark 5.7. *The operators specified in Lemma 5.5 are also well defined for the cases $\alpha = -\mu/\lambda$ and $\alpha = -2\mu/\lambda$. Thus the displacement based formulation (5.1) can be extended to these cases. The well-posedness of (5.1) for $\alpha = -2\mu/\lambda$ depends on S_h, see also Section 7.*

In a next step, we provide sufficient conditions such that the Assumptions 5.2(i)–(iii) are satisfied.

Assumption 5.8. $\operatorname{sph}_h S_{h|_K} \subset \mathcal{I} \subset S_{h|_K}$, $K \in \mathcal{T}_h$.

By construction all our cases satisfy $\mathcal{I} \subset S_{h|_K}$ and $\operatorname{sph}_h S_{h|_K} = P_0(K)\mathbf{1} \subset \mathcal{I}$. Thus Assumption 5.8 is satisfied for all our cases. We note that Assumption 5.8 guarantees that the best approximation error in the L^2-norm of the space $S_h^c \subset S_h$ is of order h.

Lemma 5.9. *Assume that Assumptions 3.1(i), 4.3(i), 5.4 and 5.8 are satisfied. If either the Assumption 3.1(ii) or $S_h = D_h$ and $0 < c_d \leq (2\mu+\alpha\lambda) \leq C_d < \infty$ hold, then the Assumptions 5.2(i)–(iii) are satisfied.*

Proof. For both cases, we have the displacement based formulation (5.1) with $Q_h\boldsymbol{\varepsilon}(\boldsymbol{v}_h) = \mathcal{C}P_{S_h^c}\boldsymbol{\varepsilon}(\boldsymbol{v}_h) + \theta(\mu,\lambda,\alpha)P_{S_h^t}\boldsymbol{\varepsilon}(\boldsymbol{v}_h)$, where

$$\theta(\mu,\lambda,\alpha) := \frac{2\mu}{1-\gamma} \quad \text{in the first case, and}$$

$$\theta(\mu,\lambda,\alpha) := \frac{4(\mu+\lambda)^2(2\mu+\alpha\lambda)}{\lambda^2+(2\mu+3\lambda)(2\mu+\alpha\lambda)} \quad \text{in the second case.}$$

Since in both cases θ is positive and bounded independently of λ from below, Assumption 4.3(i) yields

$$\begin{aligned}(Q_h\boldsymbol{\varepsilon}(\boldsymbol{v}_h),\mathcal{C}_hQ_h\boldsymbol{\varepsilon}(\boldsymbol{v}_h))_0 &= (P_{S_h^c}\boldsymbol{\varepsilon}(\boldsymbol{v}_h),\mathcal{C}P_{S_h^c}\boldsymbol{\varepsilon}(\boldsymbol{v}_h))_0+\theta(\mu,\lambda,\alpha)(P_{S_h^t}\boldsymbol{\varepsilon}(\boldsymbol{v}_h),P_{S_h^t}\boldsymbol{\varepsilon}(\boldsymbol{v}_h))_0\\ &\geq C(P_{S_h}\boldsymbol{\varepsilon}(\boldsymbol{v}_h),P_{S_h}\boldsymbol{\varepsilon}(\boldsymbol{v}_h))_0\geq C\|\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_0^2\end{aligned}$$

with a λ-independent constant C. Assumption 5.2(ii) is satisfied as a result of $Q_h = P_{S_h}$, and $\mathcal{C}_hQ_h\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\in S_h$. Now, we turn our attention to 5.2(iii). Assumption 5.8 yields $\mathrm{tr}(P_{S_h^c}-\boldsymbol{\Pi}_\mathrm{h})\boldsymbol{\varepsilon}(\boldsymbol{v}_h)=0$, and thus

$$\begin{aligned}(\mathcal{C}_hQ_h-\mathcal{C}\boldsymbol{\Pi}_h)\boldsymbol{\varepsilon}(\boldsymbol{v}_h) &= \mathcal{C}P_{S_h^c}\boldsymbol{\varepsilon}(\boldsymbol{v}_h)+\theta(\mu,\lambda,\alpha)P_{S_h^t}\boldsymbol{\varepsilon}(\boldsymbol{v}_h)-\mathcal{C}\boldsymbol{\Pi}_h\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\\ &= 2\mu(P_{S_h^c}\boldsymbol{\varepsilon}(\boldsymbol{v}_h)-\boldsymbol{\Pi}_h\boldsymbol{\varepsilon}(\boldsymbol{v}_h))+\theta(\mu,\lambda,\alpha)P_{S_h^t}\boldsymbol{\varepsilon}(\boldsymbol{v}_h).\end{aligned}$$

Since $\theta(\mu,\lambda,\alpha)$ is bounded independently of λ from above, it suffices to show that

$$\begin{aligned}\|(P_{S_h^c}-\boldsymbol{\Pi}_h)\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_0 &\leq C\|(\mathbf{Id}-\boldsymbol{\Pi}_\mathrm{h})\boldsymbol{\varepsilon}(\boldsymbol{v}_\mathrm{h})\|_0\ ,\\ \|P_{S_h^t}\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_0 &\leq C\|(\mathbf{Id}-\boldsymbol{\Pi}_\mathrm{h})\boldsymbol{\varepsilon}(\boldsymbol{v}_\mathrm{h})\|_0\,.\end{aligned}\tag{5.7}$$

To prove the first inequality in (5.7), we start with the identity

$$\|(P_{S_h^c}-\mathbf{Id})\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_0^2=((P_{S_h^c}-\mathbf{Id})\boldsymbol{\varepsilon}(\boldsymbol{v}_h),(P_{S_h^c}-\boldsymbol{\Pi}_h+\boldsymbol{\Pi}_h-\mathbf{Id})\boldsymbol{\varepsilon}(\boldsymbol{v}_h))_0.$$

By definition of $P_{S_h^c}$ and Assumption 5.8, we find that $(P_{S_h^c}-\boldsymbol{\Pi}_h)\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\in S_h^c$, and so

$$\|(P_{S_h^c}-\mathbf{Id})\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_0\leq\|(\boldsymbol{\Pi}_h-\mathbf{Id})\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_0\,.$$

Hence the first inequality in (5.7) results from the triangle inequality.

Using $\boldsymbol{\Pi}_h\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\in S_h^c$, and the orthogonality of S_h^c and S_h^t, the following upper bound is obtained:

$$\begin{aligned}\|P_{S_h^t}\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_0^2 &= (-P_{S_h^t}\boldsymbol{\varepsilon}(\boldsymbol{v}_h),(\mathbf{Id}-P_{S_h^t}-\mathbf{Id}+\boldsymbol{\Pi}_h)\boldsymbol{\varepsilon}(\boldsymbol{v}_h))_0\\ &\leq\|P_{S_h^t}\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_0\|(\mathbf{Id}-\boldsymbol{\Pi}_\mathrm{h})\boldsymbol{\varepsilon}(\boldsymbol{v}_\mathrm{h})\|_0.\end{aligned}$$

□

Now, we show that Assumptions 5.2(i)–(iii) are satisfied for all our cases. To do so we use Lemma 5.9. We recall that Assumptions 3.1(i), 4.3(i), 5.4 and 5.8 are satisfied for all our cases. Moreover Assumption 3.1(ii) holds for Case II, Case III and Case V and $D_h = S_h$ holds for Case I and Case IV.

Remark 5.10. *Lemma 5.9 provides sufficient conditions such that the displacement based formulation (5.1) is well defined and yields λ-independent optimal a priori estimates. We recall that for Cases I and IV, the choice of α is crucial. Setting $\alpha = 1$ will result in volumetric locking whereas the case $\alpha = 0$, for example, will give good numerical results. For the λ-independent estimate, the Assumption $0 < c_d \leq (2+\alpha\lambda/\mu) \leq C_d < \infty$ is essential. We remark that this condition on α is weaker than Assumption 4.3(ii).*

In the rest of this section, we show that the well known Q_1-P_0 saddlepoint problem can be analyzed as special situation of our modified Hu-Washizu formulation.

Lemma 5.11. *Case IV with $\alpha = -\frac{\mu}{\lambda}\,\frac{3\lambda+2\mu}{2\lambda+\mu}$ is equivalent to the penalized Q_1-P_0 saddlepoint problem (Stokes)*

$$\begin{array}{lclcll}2\mu(\boldsymbol{\varepsilon}(\boldsymbol{v}_h),\boldsymbol{\varepsilon}(\boldsymbol{u}_h))_0 &+& (\operatorname{div}\boldsymbol{v}_h,p_h)_0 &=& \ell(\boldsymbol{v}_h), & \boldsymbol{v}_h\in V_h\ ,\\ (\operatorname{div}\boldsymbol{u}_h,q_h)_0 &-& \frac{1}{\lambda}(p_h,q_h)_0 &=& 0, & q_h\in\widetilde{M}_h\,.\end{array}$$

Proof. By static condensation, we can eliminate the pressure from the saddle point problem and arrive at

$$2\mu(\boldsymbol{\varepsilon}(\boldsymbol{v}_h), \boldsymbol{\varepsilon}(\boldsymbol{u}_h))_0 + \lambda(\Pi_h \operatorname{div} \boldsymbol{v}_h, \Pi_h \operatorname{div} \boldsymbol{u}_h)_0 = \ell(\boldsymbol{v}_h), \quad \boldsymbol{v}_h \in V_h \,. \tag{5.8}$$

Using the given α, we show that (5.1) is equivalent to (5.8). With the given α, we get $\mathcal{C}_h \boldsymbol{\tau}_h = \mathcal{C} P_{S_h^c} \boldsymbol{\tau}_h + 2\mu P_{S_h^t} \boldsymbol{\tau}_h$. Hence

$$\begin{aligned}
(Q_h \boldsymbol{\varepsilon}(\boldsymbol{v}_h), \mathcal{C}_h Q_h \boldsymbol{\varepsilon}(\boldsymbol{u}_h))_0 &= (P_{S_h^c} \boldsymbol{\varepsilon}(\boldsymbol{v}_h), \mathcal{C} P_{S_h^c} \boldsymbol{\varepsilon}(\boldsymbol{u}_h))_0 + (P_{S_h^t} \boldsymbol{\varepsilon}(\boldsymbol{v}_h), 2\mu P_{S_h^t} \boldsymbol{\varepsilon}(\boldsymbol{u}_h))_0 \\
&= 2\mu(P_{S_h} \boldsymbol{\varepsilon}(\boldsymbol{v}_h), P_{S_h} \boldsymbol{\varepsilon}(\boldsymbol{u}_h))_0 + \lambda(\operatorname{tr} P_{S_h^c} \boldsymbol{\varepsilon}(\boldsymbol{v}_h), \operatorname{tr} P_{S_h^c} \boldsymbol{\varepsilon}(\boldsymbol{u}_h))_0 \\
&= 2\mu(P_{S_h} \boldsymbol{\varepsilon}(\boldsymbol{v}_h), P_{S_h} \boldsymbol{\varepsilon}(\boldsymbol{u}_h))_0 + \lambda(\Pi_h \operatorname{div} \boldsymbol{v}_h, \Pi_h \operatorname{div} \boldsymbol{u}_h)_0 \,.
\end{aligned}$$

Finally, the result follows by using the fact that $P_{S_h} \boldsymbol{\varepsilon}(\boldsymbol{v}_h) = \boldsymbol{\varepsilon}(\boldsymbol{v}_h)$ for Case IV. □

REMARK 5.12. *The choice* $\alpha = -\frac{\mu}{\lambda} \frac{3\lambda+2\mu}{2\lambda+\mu}$ *does not satisfy Assumption 4.3(ii) but* $0 < c_d \le (2 + \alpha\lambda/\mu) \le C_d < \infty$.

6. A priori results for the stress. The results of the previous sections show that the lack of stability resulting from the presence of a checkerboard mode is confined to the stress, and does not affect the displacement. In this section, we establish an optimal a priori result for a post-processed stress. Let $(\boldsymbol{u}_h, \boldsymbol{d}_h, \boldsymbol{\sigma}_h)$ be the unique solution of the modified Hu-Washizu formulation (3.2), and $\boldsymbol{\sigma}_h^s$ be the post-processed stress given by

$$\begin{aligned}
\boldsymbol{\sigma}_h &= \operatorname{dev_h} \boldsymbol{\sigma}_h + \operatorname{sph_h} \boldsymbol{\sigma}_h = \operatorname{dev_h} \boldsymbol{\sigma}_h + p_h \mathbf{1}, & p_h &\in \widetilde{M}_h \,, \\
\boldsymbol{\sigma}_h^s &:= \operatorname{dev_h} \boldsymbol{\sigma}_h + p_h^s \mathbf{1}, & p_h^s &:= \Pi_h^s p_h \,,
\end{aligned}$$

where Π_h^s is the L^2-projection onto the checkerboard-free subspace $\widetilde{M}_h^s$.

THEOREM 6.1. *Under the Assumptions 4.3(ii), 5.2(i)-(iii), 5.4, 5.8 and* (2.6), *the following* λ*-independent upper bound for the discretization error for the stress holds*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^s\|_0 \le Ch \|\boldsymbol{f}\|_0 \,.$$

Proof. We start by eliminating the strain from the saddlepoint problem (3.2), to obtain a modified Hellinger-Reissner two-field formulation. Using (5.3), (5.6) and the definition of $\theta(\mu, \lambda, \alpha)$, we find that

$$\begin{aligned}
\hat{a}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) - (\boldsymbol{\varepsilon}(\boldsymbol{u}_h), \boldsymbol{\tau}_h)_0 &= 0, & \boldsymbol{\tau}_h &\in S_h \,, \\
(\boldsymbol{\varepsilon}(\boldsymbol{v}_h), \boldsymbol{\sigma}_h)_0 &= \ell(\boldsymbol{v}_h), & \boldsymbol{v}_h &\in V_h \,,
\end{aligned}$$

where $\hat{a}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) := (\mathcal{C}_h^{-1} \boldsymbol{\sigma}_h, \boldsymbol{\tau}_h)_0$ with $\mathcal{C}_h^{-1} \boldsymbol{\tau}_h = \mathcal{C}^{-1} P_{S_h^c} \boldsymbol{\tau}_h + 1/\theta(\mu, \lambda, \alpha) P_{S_h^t} \boldsymbol{\tau}_h$.

We define S_h^s and $V_h^\perp$ such that the orthogonal decomposition of $S_h = S_h^s \oplus \widetilde{M}_h^u \mathbf{1}$, and $V_h = V_h^s \oplus V_h^\perp$ holds, respectively. Then, we have $(\boldsymbol{\varepsilon}(\boldsymbol{v}_h^s), q_h^u \mathbf{1})_0 = (\operatorname{div} \boldsymbol{v}_h^s, q_h^u)_0 = 0$, $\boldsymbol{v}_h^s \in V_h^s$, $q_h^u \in \widetilde{M}_h^u$. Using the orthogonality between S_h^s and $\widetilde{M}_h^u \mathbf{1}$ and observing that $(\boldsymbol{\varepsilon}(\boldsymbol{v}_h^s), p_h \mathbf{1})_0 = (\boldsymbol{\varepsilon}(\boldsymbol{v}_h^s), p_h^s \mathbf{1})_0$, we find that $(\boldsymbol{\sigma}_h^s, \boldsymbol{u}_h) \in (S_h^s \times V_h)$ satisfies

$$\begin{aligned}
\hat{a}(\boldsymbol{\sigma}_h^s, \boldsymbol{\tau}_h^s) - (\boldsymbol{\varepsilon}(\boldsymbol{u}_h), \boldsymbol{\tau}_h^s)_0 &= 0, & \boldsymbol{\tau}_h^s &\in S_h^s \,, \\
(\boldsymbol{\varepsilon}(\boldsymbol{v}_h^s), \boldsymbol{\sigma}_h^s)_0 &= \ell(\boldsymbol{v}_h^s), & \boldsymbol{v}_h^s &\in V_h^s \,.
\end{aligned} \tag{6.1}$$

The bilinear form $\hat{a}(\cdot,\cdot)$ is not uniformly in λ coercive on $S_h^s \times S_h^s$. We recall that $S_h^t \subset \operatorname{dev}_h S_h$, and decompose $\boldsymbol{\tau}_h^s$ according to $\boldsymbol{\tau}_h^s = \boldsymbol{\tau}_h^c + \boldsymbol{\tau}_h^t$, $\boldsymbol{\tau}_h^c \in S_h^c$, $\boldsymbol{\tau}_h^t \in S_h^t$, and $\boldsymbol{\tau}_h^c$ according to $\boldsymbol{\tau}_h^c = \operatorname{dev}_h \boldsymbol{\tau}_h^c + \operatorname{sph}_h \boldsymbol{\tau}_h^c$. Due to Lemma 4.2 and using the fact that $\theta(\mu,\lambda,\alpha)$ is uniformly bounded from above, we obtain

$$\begin{aligned}
\hat{a}(\boldsymbol{\tau}_h^s,\boldsymbol{\tau}_h^s) &= (\mathcal{C}^{-1}\boldsymbol{\tau}_h^c,\boldsymbol{\tau}_h^c)_0 + \frac{1}{\theta(\mu,\lambda,\alpha)}(\boldsymbol{\tau}_h^t,\boldsymbol{\tau}_h^t)_0 \\
&= (\mathcal{C}^{-1}\operatorname{dev_h}\boldsymbol{\tau}_h^c,\operatorname{dev_h}\boldsymbol{\tau}_h^c)_0 + (\mathcal{C}^{-1}\operatorname{sph_h}\boldsymbol{\tau}_h^c,\operatorname{sph_h}\boldsymbol{\tau}_h^c)_0 + \frac{1}{\theta(\mu,\lambda,\alpha)}(\boldsymbol{\tau}_h^t,\boldsymbol{\tau}_h^t)_0 \\
&\geq c(\|\operatorname{dev_h}\boldsymbol{\tau}_h^c\|_0^2 + \|\boldsymbol{\tau}_h^t\|_0^2) \geq c\|\operatorname{dev_h}(\boldsymbol{\tau}_h^c+\boldsymbol{\tau}_h^t)\|_0^2 = c\|\operatorname{dev_h}\boldsymbol{\tau}_h^s\|_0^2 \ .
\end{aligned}$$

To establish an upper bound for the norm of $\boldsymbol{\tau}_h^s$, we have to consider $\operatorname{sph_h}\boldsymbol{\tau}_h^s$ in more detail. Lemma 4.1 and the definition of S_h^s yields that $\operatorname{sph_h}\boldsymbol{\tau}_h^s = q_h^s\mathbf{1}$, $2q_h^s = \operatorname{tr}\operatorname{sph_h}\boldsymbol{\tau}_h^s \in \widetilde{M}_h^s$. Since $(V_h^s,\widetilde{M}_h^s)$ forms a stable Stokes pairing, we can find $\boldsymbol{v}_{q_h^s}^s \in V_h^s$ such that

$$(q_h^s, \operatorname{div}\boldsymbol{v}_{q_h^s}^s)_0 = 2\|q_h^s\|_0^2, \quad \|\boldsymbol{v}_{q_h^s}^s\|_1 \leq C\|q_h^s\|_0 \ .$$

and the discrete spherical part of $\boldsymbol{\tau}_h^s$ can be bounded by

$$\begin{aligned}
\|\operatorname{sph_h}\boldsymbol{\tau}_h^s\|_0^2 &= 2\|q_h^s\|_0^2 = (q_h^s,\operatorname{div}\boldsymbol{v}_{q_h^s}^s)_0 = (q_h^s\mathbf{1},\boldsymbol{\varepsilon}(\boldsymbol{v}_{q_h^s}^s))_0 = (\operatorname{sph_h}\boldsymbol{\tau}_h^s,\boldsymbol{\varepsilon}(\boldsymbol{v}_{q_h^s}^s))_0 \\
&= (\boldsymbol{\tau}_h^s,\boldsymbol{\varepsilon}(\boldsymbol{v}_{q_h^s}^s))_0 - (\operatorname{dev_h}\boldsymbol{\tau}_h^s,\boldsymbol{\varepsilon}(\boldsymbol{v}_{q_h^s}^s))_0 \\
&\leq (\boldsymbol{\tau}_h^s,\boldsymbol{\varepsilon}(\boldsymbol{v}_{q_h^s}^s))_0 + C\|\operatorname{dev_h}\boldsymbol{\tau}_h^s\|_0\|\operatorname{sph_h}\boldsymbol{\tau}_h^s\|_0 \ .
\end{aligned}$$

Applying Young's inequality, we obtain

$$\|\boldsymbol{\tau}_h^s\|_0^2 = \|\operatorname{sph_h}\boldsymbol{\tau}_h^s\|_0^2 + \|\operatorname{dev_h}\boldsymbol{\tau}_h^s\|_0^2 \leq (\boldsymbol{\tau}_h^s,\boldsymbol{\varepsilon}(\boldsymbol{v}_{q_h^s}^s))_0 + C\hat{a}(\boldsymbol{\tau}_h^s,\boldsymbol{\tau}_h^s) \ . \tag{6.2}$$

Replacing $\boldsymbol{\tau}_h^s$ by $\boldsymbol{\tau}_h^s - \boldsymbol{\sigma}_h^s$ and using the second equation in (6.1), the first term in the upper bound of (6.2) can be estimated by

$$\begin{aligned}
(\boldsymbol{\tau}_h^s - \boldsymbol{\sigma}_h^s,\boldsymbol{\varepsilon}(\boldsymbol{v}_{q_h^s-p_h^s}^s))_0 &= (\boldsymbol{\tau}_h^s - \boldsymbol{\sigma},\boldsymbol{\varepsilon}(\boldsymbol{v}_{q_h^s-p_h^s}^s))_0 + (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^s,\boldsymbol{\varepsilon}(\boldsymbol{v}_{q_h^s-p_h^s}^s))_0 \\
&\leq \|\boldsymbol{\tau}_h^s - \boldsymbol{\sigma}\|_0\|\boldsymbol{\tau}_h^s - \boldsymbol{\sigma}_h^s\|_0 \ .
\end{aligned}$$

To bound $\hat{a}(\boldsymbol{\tau}_h^s - \boldsymbol{\sigma}_h^s, \boldsymbol{\tau}_h^s - \boldsymbol{\sigma}_h^s)$, we have to use the first equation in (6.1). This gives

$$\begin{aligned}
\hat{a}(\boldsymbol{\tau}_h^s - \boldsymbol{\sigma}_h^s,\boldsymbol{\tau}_h^s - \boldsymbol{\sigma}_h^s) &= (\mathcal{C}_h^{-1}\boldsymbol{\tau}_h^s,\boldsymbol{\tau}_h^s - \boldsymbol{\sigma}_h^s)_0 - (\boldsymbol{\varepsilon}(\boldsymbol{u}_h),\boldsymbol{\tau}_h^s - \boldsymbol{\sigma}_h^s)_0 \\
&= (\mathcal{C}_h^{-1}\boldsymbol{\tau}_h^s - \mathcal{C}^{-1}\boldsymbol{\sigma},\boldsymbol{\tau}_h^s - \boldsymbol{\sigma}_h^s)_0 + (\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_h),\boldsymbol{\tau}_h^s - \boldsymbol{\sigma}_h^s)_0 \\
&\leq \|\boldsymbol{\tau}_h^s - \boldsymbol{\sigma}_h^s\|_0(\|\mathcal{C}_h^{-1}\boldsymbol{\tau}_h^s - \mathcal{C}^{-1}\boldsymbol{\sigma}\|_0 + \|\boldsymbol{\varepsilon}(\boldsymbol{u}-\boldsymbol{u}_h)\|_0) \ .
\end{aligned}$$

Combining the last two estimates with (6.2), we find that

$$\|\boldsymbol{\tau}_h^s - \boldsymbol{\sigma}_h^s\|_0 \leq C(\|\mathcal{C}^{-1}\boldsymbol{\sigma} - \mathcal{C}_h^{-1}\boldsymbol{\tau}_h^s\|_0 + \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h^s\|_0 + \|\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_h)\|_0) \ . \tag{6.3}$$

Since $S_{2h}^c \subset S_{2h} \subset S_h^s$ Assumption 5.8 guarantees that the best approximation error in the subspace $S_h^c \cap S_h^s$ satisfies $\inf_{\boldsymbol{\tau}_h^s \in S_h^c \cap S_h^s}\|\boldsymbol{\tau}_h^s - \boldsymbol{\sigma}\|_0 \leq Ch\|\boldsymbol{f}\|_0$. In terms of Theorem 5.3, the triangle inequality in combination with (6.3) yields

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^s\|_0 \leq C(\inf_{\boldsymbol{\tau}_h^s \in S_h^c \cap S_h^s}(\|\boldsymbol{\sigma} - \boldsymbol{\tau}_h^s\|_0 + \|\boldsymbol{\varepsilon}(\boldsymbol{u}) - \boldsymbol{\varepsilon}(\boldsymbol{u}_h)\|_0) \leq Ch\|\boldsymbol{f}\|_0 \ .$$

We recall that $\mathcal{C}_h$ restricted to S_h^c is identical to $\mathcal{C}$. □

7. Numerical examples. In this section, we illustrate the performance of the formulation discussed in the preceding sections in some numerical tests. In particular, we show the locking free response in the incompressible limit of the proposed formulation by comparing the results with the analytical solution and the results obtained from the standard Q_1-displacement approach. All examples are in two dimensions, and based on four-noded quadrilateral elements with standard bilinear interpolation of the displacement field. Furthermore, we assume isotropy and plane strain. The implementation is based on the finite element toolbox UG, [3]. In our two examples, we use the modified Hu-Washizu formulation (3.2) and apply static condensation.

Example 1: Cook's membrane problem. In this popular benchmark problem [22, 19, 17], we set $\Omega := \text{conv}\{(0,0),(48,44),(48,60),(0,44)\}$, where $\text{conv}\xi$ is the convex hull of the set ξ. The left boundary of the tapered panel Ω is clamped, and the right one is subjected to an in-plane shearing load of 100N along the y-direction, as shown in Figure 7.1(a). The material properties are taken to be $E = 250$ and $\nu = 0.4999$, so that a nearly incompressible response is obtained. The vertical tip displacement

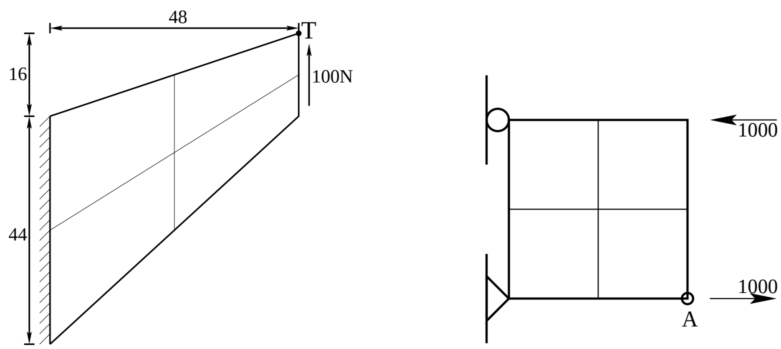

Fig. 7.1. *(a) Cook's membrane problem with initial triangulation; (b) the square beam problem with a mesh of four squares*

at the point T is computed for the different cases in Table 3.1, for different levels of uniform refinement, starting with the initial triangulation shown in Figure 7.1 (a). As can be seen from Table 7.1, the standard displacement approach and standard Hu-Washizu formulation ($\alpha = 1$) with stress and strain spaces given in Cases I and IV exhibit locking whereas all other cases show rapid convergence.

Table 7.1
Vertical tip displacement at point T, Example 1

	$\alpha = 1$		α independent			$\alpha = \frac{\mu}{4\lambda}$		$\alpha = -\frac{\mu}{4\lambda}$		$\alpha = 0$		Q_1-P_0
lev	Q_1	I	II	III	V	I	IV	I	IV	I	IV	
0	2.00	2.00	4.00	4.58	3.15	2.93	2.70	3.18	2.82	3.04	2.75	3.01
1	2.07	2.08	5.40	5.64	4.42	4.18	3.76	4.53	3.97	4.34	3.86	4.31
2	2.10	2.12	6.73	7.02	6.24	5.93	5.60	6.20	5.83	6.06	5.71	6.28
3	2.15	2.22	7.59	7.52	7.17	7.00	6.85	7.13	6.97	7.06	6.91	7.21
4	2.32	2.54	7.59	7.68	7.53	7.45	7.40	7.50	7.45	7.48	7.42	7.55
5	2.84	3.39	7.69	7.73	7.67	7.63	7.62	7.66	7.64	7.64	7.63	7.68
6	4.03	4.94	7.74	7.75	7.73	7.71	7.70	7.72	7.71	7.71	7.71	7.73

Example 2: Square beam. In the second example, we consider the domain $\Omega := (0,2) \times (0,2)$, which is fixed in the x-direction at the point $(0,2)$ and fixed in both

directions at the origin. A linearly varying horizontal force is applied in the x-direction along the boundary $x = 2$, with resultant point forces $p = 1000$ at $(2, 0)$ and $p = -1000$ at $(2, 2)$ (Figure 7.1 (b)). This problem has also been considered in [25]. In Table 7.2, we present the vertical tip displacement at the point $A := (2, 0)$ for $E = 1500$ and for different values of Poisson's ratio ν, where ν_1, ν_2, ν_3 and ν_4 are given by $0.4, 0.49, 0.499$ and 0.4999, respectively. The exact vertical displacement at A is $4(1 - \nu^2)$.

Table 7.2
Vertical tip displacement at point A, Example 2

	$\alpha = 1$		α independent			$\alpha = \frac{\mu}{4\lambda}$		$\alpha = -\frac{\mu}{4\lambda}$		$\alpha = 0$		Q_1-P_0
ν	Q_1	I	II	III	V	I	IV	I	IV	I	IV	
ν_1	2.64	2.80	3.36	4.07	3.13	3.04	2.85	3.13	2.92	3.08	2.88	3.45
ν_2	0.75	0.76	3.04	3.74	2.86	2.39	2.27	2.60	2.47	2.49	2.36	3.23
ν_3	0.09	0.09	3.00	3.70	2.83	2.30	2.19	2.53	2.41	2.41	2.29	3.20
ν_4	0.01	0.01	3.00	3.69	2.82	2.29	2.18	2.53	2.40	2.40	2.29	3.20

As in the previous example, the standard displacement approach and the standard Hu-Washizu formulation with Cases I and IV show the locking effect whereas all other cases exhibit stable behavior. In particular, Cases II, V and Q_1-P_0 give better numerical results. Furthermore, the numerical results from Case II are almost exact.

To illustrate the dependence of the numerical solution $\boldsymbol{u}_h$ on α, we show in Figure 7.2 the absolute error of the vertical tip displacement at A versus $\frac{\alpha\lambda}{2\mu}$. The left picture shows the Case IV and the right pictures shows the Case I. We set $E = 1500$ and $\nu = 0.4999$. As can be seen from Figure 7.2, the locking effect increases with $\alpha\lambda$. We note that $\mathcal{B}_\alpha$ is singular for $\alpha = \frac{-\mu}{\lambda}$ but the displacement based formulation (5.1) with Q_h and $\mathcal{C}_h$ given as in Lemma 5.5 is well defined. Therefore, we use formulation (5.1) to compute the displacement here. For $\alpha = \frac{-2\mu}{\lambda}$, we find that $\mathcal{C}_{h|_{S_h^t}} = \mathbf{0}$ and $(S_h^4)^c = S_h^3 = (S_h^3)^c$, and $(S_h^1)^c = \mathcal{I}$. Thus Case IV reduces to Case III, whereas for the Case I Assumption 5.2(i) does not hold locally for this special choice of α.

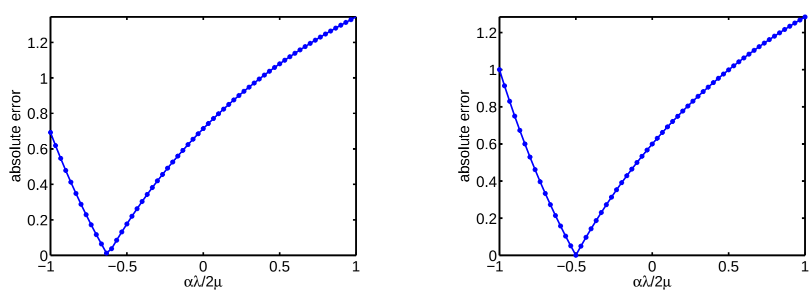

Fig. 7.2. *Error of the vertical tip displacement at A versus $\frac{\alpha\lambda}{2\mu} \in [-1, 1]$, Case IV (left) and Case I (right)*

REFERENCES

[1] U. Andelfinger and E. Ramm. EAS-elements for two-dimensional, three-dimensional, plate and shell structures and their equivalence to HR-elements. *Int. J. Numer. Meth. Engrg.*, 36:1311–1337, 1993.

[2] D.N. Arnold, L.R. Scott, and M. Vogelius. Regular inversion of the divergence operator with dirichlet boundary conditions on a polygon. *Annali Scuola Norm. Sup. Pisa, Serie 4*, 15:169–192, 1988.

[3] P. Bastian, K. Birken, K. Johannsen, S. Lang, N. Neuß, H. Rentz–Reichert, and C. Wieners. UG – a flexible software toolbox for solving partial differential equations. *Computing and Visualization in Science*, 1:27–40, 1997.

[4] D. Braess. Stability of saddle point problems with penalty. M^2AN, 30:731–742, 1996.

[5] D. Braess. Enhanced assumed strain elements and locking in membrane problems. *Comp. Meth. Appl. Mech. Engrg*, 165:155–174, 1998.

[6] D. Braess, C. Carstensen, and B. D. Reddy. Uniform convergence and a posteriori error estimators for the enhanced strain finite element method. *Numer. Math.*, 96:461–479, 2004.

[7] S.C. Brenner and L. Sung. Linear finite element methods for planar linear elasticity. *Math. Comp.*, 59:321–338, 1992.

[8] F. Brezzi and M. Fortin. *Mixed and Hybrid Finite Element Methods*. Springer-Verlag, New York, 1991.

[9] F. Brezzi and M. Fortin. A minimal stabilization procedure for mixed finite element methods. *Numer. Math.*, 89:457–491, 2001.

[10] J.K. Djoko, B.P. Lamichhane, B.D. Reddy, and B.I. Wohlmuth. Conditions for equivalence between the Hu-Washizu and related formulations, and computational behaviour in the incompressible limit. 2004. In preparation.

[11] C.A. Felippa. On the original publication of the general canonical functional of linear elasticity. *J. Appl. Mech.*, 67:217–219, 2000.

[12] B.M. Fraeijs de Veubeke. *Diffusion des inconnues hyperstatiques dans les voilures á longeron couplés*. Bull. Serv. Technique de l'Aeronautique, Impremérie Marcel Hayez, Bruxelles, 1951.

[13] B.M. Fraeijs de Veubeke. *Displacement and equilibrium models, In Stress Analysis*. Wiley, London, 1965.

[14] B.M. Fraeijs de Veubeke. Displacement and equilibrium models. *Int. J. Numer. Meth. Engrg.*, 52:287–342, 2001.

[15] V. Girault and P.-A. Raviart. *Finite Element Methods for Navier-Stokes Equations*. Springer-Verlag, Berlin, 1986.

[16] H. Hu. On some variational principles in the theory of elasticity and the theory of plasticity. *Scientia Sinica*, 4:33–54, 1955.

[17] E.P. Kasper and R.L. Taylor. A mixed-enhanced strain method. Part I: geometrically linear problems. *Comp. Struct.*, 75:237–250, 2000.

[18] E.P. Kasper and R.L. Taylor. A mixed-enhanced strain method. Part II: geometrically nonlinear problems. *Comp. Struct.*, 75:251–260, 2000.

[19] M. Küssner and B. D. Reddy. The equivalent parallelogram and parallelepiped, and their application to stabilized finite elements in two and three dimensions. *Comp. Meth. Appl. Mech. Engrg*, 190:1967–1983, 2001.

[20] T.H.H. Pian and K. Sumihara. Rational approach for assumed stress finite elements. *Int. J. Numer. Meth. Engrg.*, 20:1685–1695, 1984.

[21] B.D. Reddy and J.C. Simo. Stability and convergence of a class of enhanced strain methods. *SIAM J. Numer. Anal.*, 32:1705–1728, 1995.

[22] J.C. Simo and M.S. Rifai. A class of assumed strain methods and the method of incompatible modes. *Int. J. Numer. Meth. Engrg.*, 29:1595–1638, 1990.

[23] M. Vogelius. An analysis of the p-version of the finite element method for nearly incompressible materials. Uniformly valid, optimal error estimates. *Numer. Math.*, 41:39–53, 1983.

[24] K. Washizu. On the variational principles of elasticity and plasticity. *Report 25-18, Massachusetts Institute of Technology*, 1955.

[25] T.-X. Zhou and Y.-F. Nie. Combined hybrid approach to finite element schemes of high performance. *Int. J. Numer. Meth. Engrg.*, 51:181–202, 2001.