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Well-posedness of a model of strain gradient plasticity for plastically irrotational materials

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Abstract

The initial boundary value problem corresponding to a model of strain gradient plasticity due to [Gurtin, M., Anand, L., 2005. A theory of strain gradient plasticity for isotropic, plastically irrotational materials. Part I: Small deformations. J. Mech. Phys. Solids 53, 1624–1649] is formulated as a variational inequality, and analysed. The formulation is a primal one, in that the unknown variables are the displacement, plastic strain, and the hardening parameter. The focus of the analysis is on those properties of the problem that would ensure existence of a unique solution. It is shown that this is the case when hardening takes place. A similar property does not hold for the case of softening. The model is therefore extended by adding to it terms involving the divergence of plastic strain. For this extended model the desired property of coercivity holds, albeit only on the boundary of the set of admissible functions.

Keywords: Gradient plasticity; Elastic–plastic material B; Variational calculus C; Microstructures A

1. Introduction

Motivated by the inability of classical theories of plasticity to capture size effects and to model phenomena at the grain level to a satisfactory degree, a wide range of strain gradient theories of plasticity have emerged in recent years. A good overview of these theories

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may be found, for example, in the works by Gudmundson (2004) and by Engelen et al. (2006). In the former work it is shown that various of these theories may be subsumed within a single theoretical framework, while the latter study evaluates the effectiveness of alternative theories in modelling size effects and localisation.

Among the key gradient theories that have been proposed and studied, that due to Fleck and Hutchinson (1997) falls within the framework of higher-order continuum theories of elasticity in which the strain gradients are present and are represented in terms of both rotation and stretch gradients. In later work, Gao et al. (1999) and Huang et al. (2000) have proposed and analysed a theory whose phenomenological structure is similar to that of Fleck and Hutchinson (1997), but which differs from the latter in that the theory links the notions of plastic strain and strain gradient to the microscale phenomena of statistically stored and geometrically necessary disclocations. In an alternative approach, Huang et al. (2004) have developed a strain gradient theory that is also based on the underlying dislocation dynamics, but which preserves the basic structure of the classical theory by incorporating the strain gradient effects through the incremental plastic moduli. The relationship between gradient theories and dislocation-based mechanics at the microstructural level is given fairly extensive treatment in the literature (see, for example, Abu Al-Rub and Voyiadjis, 2006; Gurtin, 2002).

In parallel with the development of new constitutive models, theoretical aspects of the corresponding models and associated boundary value problems have also received attention. Examples include the work by de Borst and Mühlhaus (1992) who, with a view to computational analyses, consider a variational formulation for the model proposed by Dillon and Kratochvil (1970) and studied further by Mühlhaus and Aifantis (1991), as well as the investigation by Wang and Peterson (2001) of variational principles for the same model. Acharya et al. (2004) study the boundary conditions consistent with a nonlinear third-order differential equation arising from the gradient model of Acharya and Bassani (2000).

There has, on the other hand, been no detailed treatment of the well-posedness of the initial boundary value problems corresponding to gradient theories of plasticity. Such analyses are important for at least two reasons. First, they establish in a rigorous way the circumstances under which global solutions to the problem exist, and are unique. For example, gradient theories appear generally to permit solutions in the softening range, of the kind that are ruled out by classical theories. A qualitative analysis would shed light on the conditions that are sufficient to ensure that the model admits solutions in the softening regime. A second motivation is related to the need generally to develop spatially and temporally discrete approximations, for example involving finite element analyses. In such circumstances it is vital to have a proper understanding of the nature of exact solutions to the problem, as a basis upon which to determine whether approximate solutions are stable and convergent.

The purpose of this work is to carry out an analysis of the well-posedness of a model of gradient plasticity recently proposed by Gurtin and Anand (2005a) in the context of infinitesimal strains, and extended in Gurtin and Anand (2005b) to the case of finite deformations. The small-strain theory presupposes a plastic strain that has no rotational component, and is further characterised by a free energy that depends on the curl of the plastic strain. In addition to the classical momentum balance the theory also includes a microforce balance involving second-order microstresses and third-order polar microstresses.

A related finite strain theory of gradient elastoplasticity is that due to Shizawa and Zbib (1999). The theory includes kinematic hardening, and constitutive equations are obtained using the second law in the form of the Clausius–Duhem inequality.

In the analysis that follows we focus on the rate-independent case of the theory presented in Gurtin and Anand (2005a). In Section 2, the governing equations of the problem are presented. The treatment of the flow law goes beyond that in Gurtin and Anand (2005a), in that it is explicitly cast in the framework of a classical normality law associated with a von Mises-type yield condition, in order to deduce from this structure an equivalent formulation involving the dissipation function. This framework has been given an extensive treatment by Han and Reddy (1999) for the classical case. A further feature of the flow law is that it incorporates isotropic hardening in a natural way, through a scalar conjugate force which depends on the hardening variable.

This latter version of the flow law, together with the equations of equilibrium and microforce balance, permit the problem to be formulated as a variational inequality. This is the subject of Section 3. The resulting formulation is one that involves the displacements, plastic strains and their curls, and the hardening variable. It is an extension of what is referred to in Han and Reddy (1999) as the primal formulation, to distinguish it from the more traditional formulation that involves the flow law in the form of the normality law, and which has as unknown variables the displacement and stress. Various authors, for example, de Borst and Mühlhaus (1992) and Fleck and Hutchinson (2001), have commented on the suitability of what are essentially primal formulations as a basis for carrying out finite element approximations.

The analysis of well-posedness centres on the property of coercivity of the bilinear form appearing in the variational inequality, and it is shown that a unique solution exists in the case when hardening is present.

Section 4 is taken up with the issue of softening. It is shown here that coercivity is elusive when softening is present. An extension of the model, which is intended to rectify this situation by the inclusion of the divergence of plastic strain as a variable, is explored. This modification still does not lead to coercivity over the whole space on which the problem is posed, though it is shown that the bilinear form is coercive on the boundary of the set of admissible solutions, which is also where solutions corresponding to proportional loading reside. Such a result suggests that the model does indeed accommodate at least a limited amount of softening, though a rigorous proof is not possible. In contrast, the ability of the model in Mühlhaus and Aifantis (1991) to accommodate softening is clear, as is shown in the discussion that follows, and as has been alluded to Aifantis (1987) in the context of ellipticity of the equilibrium equations.

In the concluding section we comment on other theories of gradient plasticity which do not fit into the model considered here, and which will be the subject of separate treatments.

2. The governing equations for the problem

Let Ω be a bounded convex Lipschitz domain in $\mathbb{R}^3$, which is occupied by an elastoplastic body in its undeformed configuration. The boundary of Ω is denoted by $\partial\Omega$. The body is assumed to undergo infinitesimal deformations. The governing equations are those developed by Gurtin and Anand (2005a) for small-deformation gradient plasticity. The model is characterised by the inclusion of a second-order tensor $\boldsymbol{T}^{\mathrm{p}}$, the microstress, together with a third-order polar microstress $\mathbb{K}^{\mathrm{p}}$, which are conjugate respectively to the plastic strain rate $\dot{\boldsymbol{E}}^{\mathrm{p}}$ and gradient of plastic strain rate $\nabla\dot{\boldsymbol{E}}^{\mathrm{p}}$, and through which the flow law is expressed. In addition, the theory assumes zero plastic rotation. We confine attention to the important case of rate-independent plastic behaviour, though the theory in Gurtin and Anand (2005a) is valid for viscoplastic materials.

2.1. Balance laws and boundary conditions

The conventional macroscopic force balance leads to the equation of equilibrium

$$\operatorname{div}\boldsymbol{T} + \boldsymbol{b} = \boldsymbol{0} \tag{1}$$

in which $\boldsymbol{T}$ is the symmetric Cauchy stress and $\boldsymbol{b}$ is the body force. The introduction of microstresses together with a microforce balance leads to the additional balance equation

$$\boldsymbol{T}^{\mathrm{D}} = \boldsymbol{T}^{\mathrm{p}} - \operatorname{div}\mathbb{K}^{\mathrm{p}}. \tag{2}$$

Here and henceforth $\boldsymbol{T}^{\mathrm{D}}$ denotes the deviatoric part of the second-order tensor $\boldsymbol{T}$. The divergence of the third-order polar microstress is the second-order tensor with components

$$(\operatorname{div}\mathbb{K})_{ij} = K_{ijk,k} \tag{3}$$

and in which a subscript following a comma denotes partial differentiation with respect to that spatial component.

The macroscopic boundary conditions are given by

$$\boldsymbol{U} = \bar{\boldsymbol{u}} \quad \text{on } \Gamma_{\mathrm{D}}, \qquad \boldsymbol{t} := \boldsymbol{T}\boldsymbol{n} = \bar{\boldsymbol{t}} \quad \text{on } \Gamma_{\mathrm{N}}. \tag{4}$$

Here $\boldsymbol{U}$ is the displacement vector, and $\bar{\boldsymbol{u}}$ and $\bar{\boldsymbol{t}}$ are respectively prescribed displacements and surface tractions that apply on complementary parts Γ_{D} and Γ_{N} of the boundary $\partial\Omega$. The outward unit normal to $\partial\Omega$ is denoted by $\boldsymbol{n}$.

It is also necessary to prescribe microscopic boundary conditions on $\partial\Omega$; these are assumed to take the form

$$\boldsymbol{E}^{\mathrm{p}} = \boldsymbol{0} \quad \text{on } \Gamma_{\mathrm{H}}, \qquad \mathbb{K}^{\mathrm{p}}\boldsymbol{n} = \boldsymbol{0} \quad \text{on } \Gamma_{\mathrm{F}}. \tag{5}$$

The subscripts 'H' and 'F' denote 'hard' and 'free' parts of the boundary respectively, and Γ_{H} and Γ_{F} are complementary subsets of $\partial\Omega$. The homogeneous boundary conditions correspond to an assumption of null expenditure of microscopic power on the boundary (see Gurtin and Anand, 2005a, Section 8).

2.2. Constitutive theory

The constitutive equations are obtained from a free energy imbalance together with a flow law that characterises plastic behaviour.

The total strain $\boldsymbol{E}$ is given in terms of the displacement by

$$\boldsymbol{E}(\boldsymbol{U}) = \frac{1}{2}(\nabla\boldsymbol{U} + (\nabla\boldsymbol{U})^{\mathrm{T}}) \tag{6}$$

and is additively decomposed into elastic and plastic components $\boldsymbol{E}^{\mathrm{e}}$ and $\boldsymbol{E}^{\mathrm{p}}$, so that

$$\boldsymbol{E} = \boldsymbol{E}^{\mathrm{e}} + \boldsymbol{E}^{\mathrm{p}} \tag{7}$$

with the plastic strain incapable of sustaining volumetric changes. That is

$$\operatorname{tr}\boldsymbol{E}^{\mathrm{p}} = 0. \tag{8}$$

The second-order Burgers tensor $\boldsymbol{G}$ is defined to be the curl of the plastic strain: that is,

$$\boldsymbol{G} = \operatorname{curl} \boldsymbol{E}^{\mathrm{p}} \quad \text{or} \quad G_{il} = \varepsilon_{ijk} E^{\mathrm{p}}_{lk,j}, \tag{9}$$

where ε_{ijk} is the permutation symbol.

The model presented by Gurtin and Anand (2005a) is modified here in such a way as to incorporate isotropic hardening directly into the theory, in the spirit of the approach adopted in Han and Reddy (1999) (see also Menzel and Steinmann, 2000). That is, isotropic hardening is accounted for through the introduction into the free energy of a strain hardening variable γ whose conjugate g defines the hardening law. With this modification the free energy is given by

$$\psi = \hat{\psi}(\boldsymbol{E}^{\mathrm{e}}, \boldsymbol{G}, \gamma). \tag{10}$$

Linear elastic behaviour is assumed, and the dependence of ψ on $\boldsymbol{G}$ is assumed to be the most general consistent with an assumption of quadratic isotropy of the defect energy, as set out in Gurtin and Anand (2005a, Appendix). Under these conditions the free energy takes the form

$$\hat{\psi}(\boldsymbol{E}^{\mathrm{e}}, \boldsymbol{G}, \gamma) = \frac{1}{2}\boldsymbol{E}^{\mathrm{e}} : \mathscr{C}\boldsymbol{E}^{\mathrm{e}} + \frac{1}{2}\mu L^2[\beta|\boldsymbol{G}|^2 + 4(1-\beta)|\operatorname{skw}\boldsymbol{G}|^2] + \int_0^{\gamma} k(\xi)\,\mathrm{d}\xi. \tag{11}$$

For isotropic media the elasticity tensor $\mathscr{C}$ is given by

$$\mathscr{C}\boldsymbol{E} = \lambda(\operatorname{tr}\boldsymbol{E})\boldsymbol{I} + \mu(\boldsymbol{E} + \boldsymbol{E}^{\mathrm{T}}) \tag{12}$$

for any second-order tensor $\boldsymbol{E}$, where the Lamé moduli λ and μ satisfy

$$\mu > 0 \quad \text{and} \quad \lambda + 2\mu > 0$$

and $\boldsymbol{I}$ is the identity tensor. These conditions suffice for pointwise ellipticity of the elasticity tensor in the sense that there exists a constant $m_0 > 0$ such that

$$\boldsymbol{E} : \mathscr{C}\boldsymbol{E} \geqslant m_0|\boldsymbol{E}|^2 \tag{13}$$

the magnitude of the second-order tensor $\boldsymbol{E}$ being given by $|\boldsymbol{E}| = (\boldsymbol{E} : \boldsymbol{E})^{1/2}$.

In (11), L is an energetic length scale and β is a nonnegative scalar. The term $\operatorname{skw}\boldsymbol{G}$ denotes the skew part of $\boldsymbol{G}$; that is, $\operatorname{skw}\boldsymbol{G} = \frac{1}{2}(\boldsymbol{G} - \boldsymbol{G}^{\mathrm{T}})$. Finally, $k(\gamma)$ defines the isotropic hardening relation.

The local free-energy imbalance states that

$$\dot{\psi} - \boldsymbol{T} : \dot{\boldsymbol{E}}^{\mathrm{e}} - \boldsymbol{T}^{\mathrm{p}} : \dot{\boldsymbol{E}}^{\mathrm{p}} - \mathbb{K}^{\mathrm{p}} \vdots \nabla\dot{\boldsymbol{E}}^{\mathrm{p}} \leqslant 0. \tag{14}$$

Expansion of the first term and substitution of (11) gives the elastic relation

$$\boldsymbol{T} = \mathscr{C}\boldsymbol{E}^{\mathrm{e}}. \tag{15}$$

Defining the third-order tensor $\mathbb{P}$ by

$$P_{jqp} = \varepsilon_{ipq}\frac{\partial\hat{\psi}}{\partial G_{ij}} \tag{16}$$

and the quantities $\mathbb{K}^{\mathrm{p}}_{\mathrm{dis}}$ and g, conjugate respectively to $\nabla\dot{\boldsymbol{E}}^{\mathrm{p}}$ and $\dot{\gamma}$, by

$$\mathbb{K}^{\mathrm{p}}_{\mathrm{dis}} = \mathbb{K}^{\mathrm{p}} - \mathbb{P} \quad \text{and} \quad g = -\frac{\partial\hat{\psi}}{\partial\gamma} = -k(\gamma) \tag{17}$$

the dissipation inequality becomes

$$\boldsymbol{T}^{\mathrm{p}} : \dot{\boldsymbol{E}}^{\mathrm{p}} + \mathbb{K}^{\mathrm{p}}_{\mathrm{dis}} \,\vdots\, \nabla\dot{\boldsymbol{E}}^{\mathrm{p}} + g\dot{\gamma} \geqslant 0. \tag{18}$$

Remark. In Gurtin and Anand (2005a), it is the symmetric-deviatoric part of $\mathbb{P}$ that enters (17). It will be seen later that this specialisation is not necessary.

2.3. The flow law

The dissipation inequality (18) sets the scene for the development of a flow theory of plasticity along the lines of that described in Han and Reddy (1999), and which has its antecedents in the generalised standard theory of Halphen and Nguyen (1975). The treatment that follows borrows heavily from the presentation in Han and Reddy (1999), and extends that presentation to the case of gradient plasticity for the model under consideration.

Guided by (18), we introduce the triple $\boldsymbol{\Sigma} := (\boldsymbol{T}^{\mathrm{p}}, \mathbb{K}^{\mathrm{p}}_{\mathrm{dis}}, g)$ of generalised stresses, and the conjugate triple $\boldsymbol{\Gamma} := (\boldsymbol{E}^{\mathrm{p}}, \nabla\boldsymbol{E}^{\mathrm{p}}, \gamma)$ of internal variables, so that the dissipation inequality reads

$$\boldsymbol{\Sigma} \,\vdots\, \dot{\boldsymbol{\Gamma}} \geqslant 0. \tag{19}$$

The notation $\vdots$ denotes the inner product, evaluated componentwise. Next we introduce the yield function $\varphi(\boldsymbol{\Sigma})$, defined by

$$\varphi(\boldsymbol{\Sigma}) = \sqrt{|\boldsymbol{T}^{\mathrm{p}}|^2 + \ell^{-2}|\mathbb{K}^{\mathrm{p}}_{\mathrm{dis}}|^2} + g - c_0. \tag{20}$$

Here ℓ is a dissipative length scale and c_0 is related to the initial yield stress in uniaxial tension.

Remark. It is possible to construct gradient theories which accommodate multiple length scales. This is the core of the model proposed by Fleck and Hutchinson (2001), whose theory extends that of Aifantis (1987) to one in which up to three length scales are accommodated. In principle a similar extension could be carried out of the theory being studied in this work; we confine attention to the model proposed in Gurtin and Anand (2005a), which has a single dissipative length scale, since it contains the essential features relevant to a study of well-posedness.

The region $\mathscr{E}$ of admissible generalised stresses is given by

$$\mathscr{E} = \{\boldsymbol{\Sigma} \mid \varphi(\boldsymbol{\Sigma}) \leqslant 0\} \tag{21}$$

so that the elastic region corresponds to the interior of $\mathscr{E}$, and the yield surface its boundary.

A normality law is assumed, with flow governed by the inclusion

$$\dot{\boldsymbol{\Gamma}} \in N_{\mathscr{E}}(\boldsymbol{\Sigma}). \tag{22}$$

Here $N_{\mathscr{E}}(\boldsymbol{\Sigma})$ is the normal cone to $\mathscr{E}$ at $\boldsymbol{\Sigma}$, and (22) is equivalent to the statement that the generalised strains $\dot{\boldsymbol{\Gamma}}$ satisfy

$$(\bar{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}) \,\vdots\, \dot{\boldsymbol{\Gamma}} \leqslant 0 \quad \text{for all } \bar{\boldsymbol{\Sigma}} \in \mathscr{E}. \tag{23}$$

When the yield surface is smooth, as is the case here, then (22) becomes the classical normality relation or flow law

$$\dot{\boldsymbol{\Gamma}} = \lambda \frac{\partial \varphi}{\partial \boldsymbol{\Sigma}}, \tag{24}$$

the derivatives being taken componentwise, and in which the scalar multiplier λ satisfies the Karush–Kuhn–Tucker conditions

$$\lambda \geqslant 0, \quad \varphi(\boldsymbol{\Sigma}) \leqslant 0, \quad \lambda\varphi(\boldsymbol{\Sigma}) = 0.$$

When $\varphi = 0$ we have $\sqrt{|\boldsymbol{T}^{\mathrm{p}}|^2 + \ell^{-2}|\mathbb{K}^{\mathrm{p}}_{\mathrm{dis}}|^2} = c_0 - g$ and (24) yields, upon differentiation,

$$\begin{aligned}
\dot{\boldsymbol{E}}^{\mathrm{p}} &= \lambda \frac{\boldsymbol{T}^{\mathrm{p}}}{c_0 - g},\\
\nabla \dot{\boldsymbol{E}}^{\mathrm{p}} &= \lambda \ell^{-2} \frac{\mathbb{K}^{\mathrm{p}}_{\mathrm{dis}}}{c_0 - g},\\
\dot{\gamma} &= \lambda.
\end{aligned} \tag{25}$$

From $(25)_1$ and $(25)_2$ it is readily seen that

$$d^{\mathrm{p}}(\dot{\boldsymbol{E}}^{\mathrm{p}}) := \sqrt{|\dot{\boldsymbol{E}}^{\mathrm{p}}|^2 + \ell^2 |\nabla \dot{\boldsymbol{E}}^{\mathrm{p}}|^2} = \dot{\gamma} = \lambda, \tag{26}$$

$d^{\mathrm{p}}(\dot{\boldsymbol{E}}^{\mathrm{p}})$ being an effective generalised plastic strain rate. We note also from (26) the interpretation of the hardening parameter γ, as being the quantity whose rate of change coincides with the generalised plastic strain rate.

The normality law may be inverted by noting first that the region of admissible generalised stresses is a convex set. Thus (22) may be stated *equivalently* as the condition

$$\boldsymbol{\Sigma} \in \partial D(\dot{\boldsymbol{\Gamma}}), \tag{27}$$

in which D is the dissipation function, defined by

$$D(\boldsymbol{M}) = \sup\{\boldsymbol{\Sigma} \,\vdots\, \boldsymbol{M} | \boldsymbol{\Sigma} \in \mathscr{E}\} \tag{28}$$

and where $\boldsymbol{M} = (\boldsymbol{Q}, \nabla \boldsymbol{Q}, \xi)$ is an arbitrary triple. In (27), $\partial D(\dot{\boldsymbol{\Gamma}})$ denotes the subdifferential of D evaluated at $\dot{\boldsymbol{\Gamma}}$. Thus, written out in full, (27) is the statement

$$D(\boldsymbol{M}) \geqslant D(\dot{\boldsymbol{\Gamma}}) + \boldsymbol{\Sigma} \,\vdots\, (\boldsymbol{M} - \dot{\boldsymbol{\Gamma}}) \quad \text{for } \boldsymbol{\Sigma} \in \mathscr{E}. \tag{29}$$

At those points at which D is differentiable (27) reads

$$\boldsymbol{\Sigma} = \left.\frac{\partial D}{\partial \boldsymbol{M}}\right|_{\dot{\Gamma}}. \tag{30}$$

Following the procedure used in Han and Reddy (1999) (see Section 4.2) we evaluate the dissipation function corresponding to the case of the von Mises yield function with isotropic hardening, and with gradient terms present. That is, we need to maximise the quantity

$$\boldsymbol{\Sigma} \,\vdots\, \boldsymbol{M} = \boldsymbol{Q} : \boldsymbol{T}^{\mathrm{p}} + \nabla \boldsymbol{Q} \,\vdots\, \mathbb{K}^{\mathrm{p}}_{\mathrm{dis}} + g\xi \tag{31}$$

over all generalised stresses that satisfy

$$\sqrt{|\boldsymbol{T}^{\mathrm{p}}|^2 + \ell^{-2}|\mathbb{K}^{\mathrm{p}}_{\mathrm{dis}}|^2} \leqslant c_0 - g. \tag{32}$$

The largest value is achieved in fact when $\sqrt{|\boldsymbol{T}^{\mathrm{p}}|^2 + \ell^{-2}|\mathbb{K}^{\mathrm{p}}_{\mathrm{dis}}|^2} = c_0 - g$,and when the generalised vectors $(\boldsymbol{Q}, \nabla\boldsymbol{Q})$ and $(\boldsymbol{T}^{\mathrm{p}}, \mathbb{K}^{\mathrm{p}})$ are co-directional. When this is the case we have

$$\begin{aligned}
\boldsymbol{\Sigma} \,\vdots\, \boldsymbol{M} &= [\boldsymbol{T}^{\mathrm{p}} : \boldsymbol{Q} + (\ell^{-1}\mathbb{K}^{\mathrm{p}}_{\mathrm{dis}} \,\vdots\, (\ell\nabla\boldsymbol{Q})] + g\xi \\
&= \sqrt{|\boldsymbol{T}^{\mathrm{p}}|^2 + \ell^{-2}|\mathbb{K}^{\mathrm{p}}_{\mathrm{dis}}|^2}\sqrt{|\boldsymbol{Q}|^2 + \ell^2|\nabla\boldsymbol{Q}|^2} + g\xi \\
&= (c_0 - g)\sqrt{|\boldsymbol{Q}|^2 + \ell^2|\nabla\boldsymbol{Q}|^2} + g\xi \\
&= (\xi - \sqrt{|\boldsymbol{Q}|^2 + \ell^2|\nabla\boldsymbol{Q}|^2})g + c_0\sqrt{|\boldsymbol{Q}|^2 + \ell^2|\nabla\boldsymbol{Q}|^2}.
\end{aligned} \tag{33}$$

It now remains to maximise over all admissible values of g. First we note that we must have $g \leqslant c_0$, otherwise the expression (32) does not make sense. Hence, when $\xi < \sqrt{|\boldsymbol{Q}|^2 + \ell^2|\nabla\boldsymbol{Q}|^2}$ the first term in the last line of (33) can be made arbitrarily large by choosing g to be arbitrarily large and negative. On the other hand, when $\xi \geqslant \sqrt{|\boldsymbol{Q}|^2 + \ell^2|\nabla\boldsymbol{Q}|^2}$ the maximum value of $c_0\xi$ occurs when $g = c_0$. Summarising, we have

$$D(\boldsymbol{M}) = \begin{cases} c_0\xi & \text{if } \sqrt{|\boldsymbol{Q}|^2 + \ell^2|\nabla\boldsymbol{Q}|^2} \leqslant \xi, \\ \infty & \text{otherwise.} \end{cases} \tag{34}$$

The inversion leading to (30) is most easily carried out directly from (25), and leads to the set of expressions

Table 1
Summary of the models for the classical and gradient (Gurtin and Anand, 2005a) theories

Free energy	
Classical	$\hat{\psi}(\boldsymbol{E}^{\mathrm{e}}, \gamma) = \hat{\psi}^{\mathrm{e}}(\boldsymbol{E}^{\mathrm{e}}) + \hat{\psi}^{\gamma}(\gamma)$
Gradient	$\hat{\psi}(\boldsymbol{E}^{\mathrm{e}}, \boldsymbol{G}, \gamma) = \hat{\psi}^{\mathrm{e}}(\boldsymbol{E}^{\mathrm{e}}) + \hat{\psi}^{G}(\boldsymbol{G}) + \hat{\psi}^{\gamma}(\gamma)$
Elastically stored energy	$\hat{\psi}^{\mathrm{e}}(\boldsymbol{E}^{\mathrm{e}}) = \frac{1}{2}\boldsymbol{E}^{\mathrm{e}} : \mathscr{C}\boldsymbol{E}^{\mathrm{e}}$
Isotropic part of the free energy	$\hat{\psi}^{\gamma}(\gamma) = \int_0^{\gamma} k(\xi)\,\mathrm{d}\xi$
Gradient part of the free energy	$\hat{\psi}^{G}(\boldsymbol{G}) = \frac{1}{2}\mu L^2[\beta|\boldsymbol{G}|^2 + 4(1-\beta)|\mathrm{skw}\,\boldsymbol{G}|^2]$
Yield condition of von Mises type	
Classical	$\varphi(\boldsymbol{\Sigma}) = |\boldsymbol{T}^{\mathrm{D}}|^2 + g - c_0$
Gradient	$\varphi(\boldsymbol{\Sigma}) = \sqrt{|\boldsymbol{T}^{\mathrm{p}}|^2 + \ell^{-2}|\mathbb{K}^{\mathrm{p}}_{\mathrm{dis}}|^2} + g - c_0$
Macroscopic Cauchy stress	$\boldsymbol{T} = \mathscr{C}\boldsymbol{E}^{\mathrm{e}}$
Isotropic hardening parameter (linear)	$g = -k\gamma$
Flow rule	
Classical	$\boldsymbol{T}^{\mathrm{p}} = (c_0 - g)\frac{\dot{\boldsymbol{E}}^{\mathrm{p}}}{|\dot{\boldsymbol{E}}^{\mathrm{p}}|}$ $|\dot{\boldsymbol{E}}^{\mathrm{p}}| = \dot{\gamma} = \lambda$
Gradient	$\boldsymbol{T}^{\mathrm{p}} = (c_0 - g)\frac{\dot{\boldsymbol{E}}^{\mathrm{p}}}{d^{\mathrm{p}}}, \quad \mathbb{K}^{\mathrm{p}}_{\mathrm{dis}} = (c_0 - g)\ell^2\frac{\nabla\dot{\boldsymbol{E}}^{\mathrm{p}}}{d^{\mathrm{p}}}$ $d^{\mathrm{p}}(\dot{\boldsymbol{E}}^{\mathrm{p}}) = \sqrt{|\dot{\boldsymbol{E}}^{\mathrm{p}}|^2 + \ell^2|\nabla\dot{\boldsymbol{E}}^{\mathrm{p}}|^2} = \dot{\gamma} = \lambda$

$$\begin{aligned} \boldsymbol{T}^{\mathrm{p}} &= (c_0 - g)\frac{\dot{\boldsymbol{E}}^{\mathrm{p}}}{d^{\mathrm{p}}}, \\ \mathbb{K}^{\mathrm{p}}_{\mathrm{dis}} &= (c_0 - g)\ell^2\frac{\nabla\dot{\boldsymbol{E}}^{\mathrm{p}}}{d^{\mathrm{p}}}, \end{aligned} \tag{35}$$

with d^p defined in (26). These should be compared with the analogous expressions given in Gurtin and Anand (2005a, Eqs. (6.12)). The key difference between those expressions and (35) lies in the flow law for $\mathbb{K}^{\mathrm{p}}_{\mathrm{dis}}$, in which the hardening-related coefficient $(c_0 - g)$ is replaced there by c_0, which is effectively the initial yield stress.

Before discussing the variational problem, for convenience and ease of comparison we summarise in Table 1 the key elements of the classical and gradient models.

3. The variational problem

The analysis that follows is rendered much simpler if we convert the problem to one involving homogeneous displacement boundary conditions. To do this, we construct a function $\bar{\boldsymbol{u}}$ over the whole domain Ω with the property that this function coincides with the prescribed displacement $\bar{\boldsymbol{u}}$ on Γ_{D}. This is always possible provided that the prescribed boundary displacement is, for example, piecewise smooth. Then setting $\boldsymbol{u} = \boldsymbol{U} - \bar{\boldsymbol{u}}$ where $\boldsymbol{U}$ is the actual displacement field appearing in (4), the elastic constitutive equation (15) now becomes

$$\boldsymbol{T} = \mathscr{C}(\boldsymbol{E}(\boldsymbol{U}) - \boldsymbol{E}^{\mathrm{p}}) = \mathscr{C}(\boldsymbol{E}(\boldsymbol{u}) - \boldsymbol{E}^{\mathrm{p}}) + \mathscr{C}\boldsymbol{E}(\bar{\boldsymbol{u}}) \tag{36}$$

in which the last term amounts to a prestress. Furthermore, the macroscopic boundary conditions (4) now read

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_{\mathrm{D}} \quad \text{and} \quad \boldsymbol{T}\boldsymbol{n} = \bar{\boldsymbol{t}} \quad \text{on } \Gamma_{\mathrm{N}}. \tag{37}$$

We will pose the variational problem with the displacement $\boldsymbol{u}$ rather than $\boldsymbol{U}$ as one of the unknown variables.

To obtain the variational form of the problem we proceed formally, starting with the equilibrium equation (1). Taking the inner product of this equation with $\boldsymbol{v} - \dot{\boldsymbol{u}}$ where $\boldsymbol{v}$ is an arbitrary function which satisfies the Dirichlet boundary condition $(37)_1$, integrating over Ω, and then integrating by parts, also substituting (36) for $\boldsymbol{T}$, we obtain

$$\begin{aligned} \int_\Omega \mathscr{C}(\boldsymbol{E}(\boldsymbol{u}) - \boldsymbol{E}^{\mathrm{p}}) : (\boldsymbol{E}(\boldsymbol{v}) - \dot{\boldsymbol{u}})\,\mathrm{d}x &= \int_\Omega \boldsymbol{b}\cdot(\boldsymbol{v} - \dot{\boldsymbol{u}})\,\mathrm{d}x + \int_{\Gamma_{\mathrm{N}}} \boldsymbol{t}\cdot(\boldsymbol{v} - \dot{\boldsymbol{u}})\,\mathrm{d}s \\ &\quad + \int_\Omega \mathscr{C}\boldsymbol{E}(\bar{\boldsymbol{u}}) : \boldsymbol{E}(\boldsymbol{v})\,\mathrm{d}x. \end{aligned} \tag{38}$$

Next, expanding (29) and integrating over Ω we have

$$\begin{aligned} \int_\Omega D(\boldsymbol{Q}, \nabla\boldsymbol{Q}, \xi)\,\mathrm{d}x \geqslant \int_\Omega D(\dot{\boldsymbol{E}}^{\mathrm{p}}, \nabla\dot{\boldsymbol{E}}^{\mathrm{p}}, \dot{\gamma})\,\mathrm{d}x + \int_\Omega \boldsymbol{T}^{\mathrm{p}} : (\boldsymbol{Q} - \dot{\boldsymbol{E}}^{\mathrm{p}})\,\mathrm{d}x \int_\Omega \mathbb{K}^{\mathrm{p}}_{\mathrm{dis}} \,\vdots\, \nabla(\boldsymbol{Q} - \dot{\boldsymbol{E}}^{\mathrm{p}})\,\mathrm{d}x \\ + \int_\Omega g(\xi - \dot{\gamma})\,\mathrm{d}x. \end{aligned} \tag{39}$$

Using (2) and (17), and assuming that both $\boldsymbol{E}^{\mathrm{p}}$ and the arbitrary plastic strain $\boldsymbol{Q}$ satisfy the hard microscopic boundary conditions and $\mathbb{K}^{\mathrm{p}}$ the free boundary condition, the third term on the right-hand side of (39) becomes, after integrating by parts,

$$\begin{aligned}\int_\Omega \mathbb{K}^{\mathrm{p}}_{\mathrm{dis}} \,\vdots\, \nabla(\boldsymbol{Q} - \dot{\boldsymbol{E}}^{\mathrm{p}})\,\mathrm{d}x &= \int_\Omega (\mathbb{K}^{\mathrm{p}} - \mathbb{P}) \,\vdots\, (\nabla\boldsymbol{Q} - \nabla\dot{\boldsymbol{E}}^{\mathrm{p}})\,\mathrm{d}x \\ &= \int_\Omega \left[-\mathrm{div}\,\mathbb{K}^{\mathrm{p}} : (\boldsymbol{Q} - \dot{\boldsymbol{E}}^{\mathrm{p}}) - \mathbb{P}\vdots\nabla(\boldsymbol{Q} - \dot{\boldsymbol{E}}^{\mathrm{p}})\right]\mathrm{d}x \\ &= \int_\Omega \left[(\boldsymbol{T}^{\mathrm{D}} - \boldsymbol{T}^{\mathrm{p}}) : (\boldsymbol{Q} - \dot{\boldsymbol{E}}^{\mathrm{p}}) - \mathbb{P}\vdots\nabla(\boldsymbol{Q} - \dot{\boldsymbol{E}}^{\mathrm{p}})\right]\mathrm{d}x. \end{aligned} \tag{40}$$

Next, we return to the definition (16) of $\mathbb{P}$ and note that

$$\mathbb{P} \,\vdots\, \nabla\boldsymbol{Q} = \frac{\partial\hat{\psi}}{\partial\boldsymbol{G}} : \mathrm{curl}\,\boldsymbol{Q}.$$

Substitution of this result and (40) in (39) leads to the inequality

$$\begin{aligned}\int_\Omega D(\boldsymbol{Q}, \nabla\boldsymbol{Q}, \xi)\,\mathrm{d}x \geqslant \int_\Omega D(\dot{\boldsymbol{E}}^{\mathrm{p}}, \nabla\dot{\boldsymbol{E}}^{\mathrm{p}}, \dot{\gamma})\,\mathrm{d}x + \int_\Omega \boldsymbol{T} : (\boldsymbol{Q} - \dot{\boldsymbol{E}}^{\mathrm{p}})\,\mathrm{d}x \\ - \int_\Omega \frac{\partial\hat{\psi}}{\partial\boldsymbol{G}} : \mathrm{curl}\,(\boldsymbol{Q} - \dot{\boldsymbol{E}}^{\mathrm{p}})\,\mathrm{d}x + \int_\Omega g(\xi - \dot{\gamma})\,\mathrm{d}x. \end{aligned} \tag{41}$$

Remark

1. Since the plastic strains are trace-free, the stress $\boldsymbol{T}$ in (41) may be replaced by its deviator $\boldsymbol{T}^{\mathrm{D}}$.
2. Notice that the weak form of the problem does not require a constitutive equation for the microstress $\boldsymbol{T}^{\mathrm{p}}$, which is eliminated through the use of the microforce balance.

Next, we define the bilinear form

$$\begin{aligned} a(\boldsymbol{w}, \boldsymbol{z}) = \int_\Omega \mathscr{C}(\boldsymbol{E}(\boldsymbol{u}) - \boldsymbol{E}^{\mathrm{p}}) : (\boldsymbol{E}(\boldsymbol{v}) - \boldsymbol{Q})\,\mathrm{d}x + \int_\Omega k(\gamma)\xi\,\mathrm{d}x \\ + \mu L^2 \int_\Omega [\beta\,\mathrm{curl}\,\boldsymbol{E}^{\mathrm{p}} : \mathrm{curl}\,\boldsymbol{Q} + 4(1-\beta)(\mathrm{skw}\,\mathrm{curl}\,\boldsymbol{E}^{\mathrm{p}}) : (\mathrm{skw}\,\mathrm{curl}\,\boldsymbol{Q})]\mathrm{d}x. \end{aligned} \tag{42}$$

$\boldsymbol{w} = (\boldsymbol{u}, \boldsymbol{E}^{\mathrm{p}}, \gamma)$ is the solution and $\boldsymbol{z} = (\boldsymbol{v}, \boldsymbol{Q}, \xi)$ is an arbitrary set of displacements and generalised strains. We also define the functional j and linear functional ℓ by

$$\begin{aligned} j(\boldsymbol{z}) &= \int_\Omega D(\boldsymbol{Q})\,\mathrm{d}x, \\ \ell(\boldsymbol{z}) &= \int_\Omega \boldsymbol{b}\cdot\boldsymbol{v}\,\mathrm{d}x + \int_{\Gamma_{\mathrm{N}}} \boldsymbol{t}\cdot\boldsymbol{v}\,\mathrm{d}s + \int_\Omega \mathscr{C}\boldsymbol{E}(\bar{\boldsymbol{u}}) : \boldsymbol{E}(\boldsymbol{v})\,\mathrm{d}x. \end{aligned} \tag{43}$$

Then by adding the weak form of the equilibrium equation (38) to the inequality (41) we obtain the statement of the problem in the form of a variational inequality

$$a(\boldsymbol{w}(t), \boldsymbol{z} - \dot{\boldsymbol{w}}(t)) + j(\boldsymbol{z}) - j(\dot{\boldsymbol{w}}(t)) - \ell(\boldsymbol{z} - \dot{\boldsymbol{w}}(t)) \geqslant 0 \tag{44}$$

that is, we are required to find $\boldsymbol{u}, \boldsymbol{E}^{\mathrm{p}}, \gamma$, all functions of time, which satisfy (44) for any arbitrary $\boldsymbol{v}$, $\boldsymbol{Q}$ and ξ.

In order to study the well-posedness of the problem (44), that is, the conditions under which a unique solution exists, it is necessary to define the spaces of functions in which solutions are sought. First we introduce the space of displacements $\mathscr{V}$; this is defined to be the space of all functions with finite energy, in the sense that

$$\int_{\Omega} |\boldsymbol{E}(\boldsymbol{v})|^2 \, \mathrm{d}x \text{ is finite} \tag{45}$$

for all admissible displacements, and which furthermore satisfy the homogeneous boundary condition on Γ_{D}; that is,

$$\mathscr{V} = \{\boldsymbol{v} \mid \boldsymbol{v} \text{ has finite energy, } \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_{\mathrm{D}}\}. \tag{46}$$

This is a Hilbert space when endowed with the inner product

$$(\boldsymbol{u}, \boldsymbol{v})_{\mathscr{V}} = \int_{\Omega} \boldsymbol{E}(\boldsymbol{u}) : \boldsymbol{E}(\boldsymbol{v}) \, \mathrm{d}x \tag{47}$$

and associated norm

$$\|\boldsymbol{v}\|_{\mathscr{V}} = (\boldsymbol{v}, \boldsymbol{v})_{\mathscr{V}}^{1/2} . \tag{48}$$

That the quantity in (48) is indeed a norm may be verified through the use of Korn's inequality (see, for example, Han and Reddy, 1999) which states that there exists a constant $c_K > 0$ such that

$$c_K \int_{\Omega} (|\boldsymbol{v}|^2 + |\nabla \boldsymbol{v}|^2) \, \mathrm{d}x \leqslant \int_{\Omega} |\boldsymbol{E}(\boldsymbol{v})|^2 \, \mathrm{d}x. \tag{49}$$

Remark. If the boundary conditions are purely of the traction kind, so that $\Gamma_{\mathrm{N}} = \partial\Omega$, then it becomes necessary to add further conditions to the specification of the space of admissible displacements, to rule out rigid body translations and rotations. Specifically, we then require in addition that

$$\int_{\Omega} \boldsymbol{u} \, \mathrm{d}x = \boldsymbol{0} \quad \text{and} \quad \int_{\Omega} \boldsymbol{u} \times \boldsymbol{x} \, \mathrm{d}x = \boldsymbol{0}. \tag{50}$$

Furthermore, the loading is required to satisfy the compatibility condition that the net total force and couple acting on the body be zero: that is,

$$\int_{\Omega} \boldsymbol{b} \, \mathrm{d}x + \int_{\Gamma_{\mathrm{N}}} \boldsymbol{t} \, \mathrm{d}s = \boldsymbol{0} \quad \text{and} \quad \int_{\Omega} \boldsymbol{b} \times \boldsymbol{x} \, \mathrm{d}x + \int_{\Gamma_{\mathrm{N}}} \boldsymbol{t} \times \boldsymbol{x} \, \mathrm{d}s = \boldsymbol{0}. \tag{51}$$

With these modifications the analysis then proceeds in much the same way as will be presented below.

We will also need spaces of admissible plastic strains and hardening parameters. These are defined in an analogous way by

$$\begin{aligned} \mathscr{Q} &= \left\{ \boldsymbol{Q} \middle| \int_{\Omega} |\nabla \boldsymbol{Q}|^2 \, \mathrm{d}x \text{ is finite, } Q_{ji} = Q_{ij}, \ \operatorname{tr} \boldsymbol{Q} = 0, \ \boldsymbol{Q} = \boldsymbol{0} \text{ on } \Gamma_{\mathrm{H}} \right\}, \\ \mathscr{M} &= \left\{ \xi \middle| \int_{\Omega} \xi^2 \, \mathrm{d}x \text{ is finite} \right\}. \end{aligned} \tag{52}$$

In the definition of $\mathscr{Q}$ we require that admissible plastic strains are symmetric and satisfy the conditions of plastic incompressibility and the boundary condition (5). Both $\mathscr{Q}$ and $\mathscr{M}$ are Hilbert spaces with norms

$$\|\boldsymbol{Q}\|_{\mathscr{Q}}^2 = \int_\Omega |\nabla \boldsymbol{Q}|^2 \,\mathrm{d}x, \quad \|\xi\|_{\mathscr{M}}^2 = \int_\Omega \xi^2 \,\mathrm{d}x. \tag{53}$$

That $\|\cdot\|_{\mathscr{Q}}$ is a norm follows from the Poincaré inequality (Evans, 1998), according to which there exists a constant $c_P > 0$ such that

$$c_P \int_\Omega |\boldsymbol{Q}|^2 \,\mathrm{d}x \leqslant \int_\Omega |\nabla \boldsymbol{Q}|^2 \,\mathrm{d}x \tag{54}$$

for all functions $\boldsymbol{Q}$ having finite energy, in the sense of $(52)_1$.

Remark. The special case in which the microscopic boundary condition is only the free condition $(5)_2$ is accommodated by the same choice of admissible spaces for the plastic strains, and no further condition of the kind (50) is needed for this case. This is because the bilinear form $a(\cdot,\cdot)$ contains terms involving both $\boldsymbol{E}^{\mathrm{p}}$ and its derivatives, unlike the displacement, which occurs only in the form of derivatives.

Since the set of unknowns will be the displacement $\boldsymbol{u}$, the plastic strain $\boldsymbol{E}^{\mathrm{p}}$, and the hardening parameter γ, for convenience we define the product space $\mathscr{Z}$ by

$$\mathscr{Z} = \mathscr{V} \times \mathscr{Q} \times \mathscr{M}\ . \tag{55}$$

Then $\mathscr{Z}$ is a Hilbert space with norm $\|\cdot\|_{\mathscr{Z}}$ defined by

$$\|\boldsymbol{z}\|_{\mathscr{Z}} = (\|\boldsymbol{v}\|_{\mathscr{V}}^2 + \|\boldsymbol{Q}\|_{\mathscr{Q}}^2 + \|\xi\|_{\mathscr{M}}^2)^{1/2} \qquad \forall \boldsymbol{z} = (\boldsymbol{v}, \boldsymbol{Q}, \xi) \in \mathscr{Z}. \tag{56}$$

We will have to work within the subset of unknowns for which the dissipation (34) is finite. Thus we define the subset $\mathscr{W}$ of $\mathscr{Z}$ by

$$\mathscr{W} = \left\{ (\boldsymbol{v}, \boldsymbol{Q}, \xi) \in \mathscr{Z} : \sqrt{|\boldsymbol{Q}|^2 + \ell^2 |\nabla \boldsymbol{Q}|^2} \leqslant \xi \text{ in } \Omega \right\}. \tag{57}$$

We are now in a position to formulate the *variational problem*.

Problem 1. Find $\boldsymbol{w}(t) = (\boldsymbol{u}(t), \boldsymbol{E}^{\mathrm{p}}(t), \gamma(t))$ in $\mathscr{Z}$ which satisfies the variational inequality (44) for all $\boldsymbol{z} = (\boldsymbol{v}, \boldsymbol{Q}, \xi)$ in $\mathscr{W}$.

Remark. The presence of terms involving curls in (42) is reminiscent of problems that arise in electromagnetic theory (see, for example, Duvaut and Lions, 1976), and suggests that the space of plastic strains be defined as those functions whose curls are also square-integrable. However, the set $\mathscr{W}$ in which solutions are sought is defined in terms of plastic strains and their *gradients*, and this makes it essential that the relevant space be defined in terms of $\nabla \boldsymbol{Q}$ and not merely $\operatorname{curl} \boldsymbol{Q}$, since a square-integrable gradient implies a square-integrable curl, but not vice versa.

We come now to the main question, viz. that concerning the well-posedness of Problem 1. We will make use of an abstract result presented in Han and Reddy (1999, Sections 7.2 and 7.3), in terms of which Problem 1 has a unique solution for all t, provided that the bilinear form and functionals are all continuous, the functional j is convex and positively homogeneous, and the bilinear form $a(\cdot,\cdot)$ is coercive on $\mathscr{W}$. The properties of continuity, convexity and positive homogeneity are easily demonstrated, and the question of well-posedness is thus reduced to one of establishing the conditions under which $a(\cdot,\cdot)$ is coercive. That is, we require that there exist a constant $\alpha > 0$ such that

$$a(z,z) \geqslant \alpha\|z\|_Z^2 \quad \text{for all } z \text{ in } \mathscr{W}. \tag{58}$$

It will be seen that the property of coercivity is intimately linked to the hardening properties of the material. For convenience we assume that hardening behaviour is linear, so that the function $k(\gamma)$ in (42) is given by

$$k(\gamma) = k_1\gamma. \tag{59}$$

Hardening thus corresponds to the case $k_1 > 0$, while $k_1 = 0$ gives the case of perfect plasticity, and $k_1 < 0$ represents softening behaviour. The extension to nonlinear hardening is carried out without difficulty.

We now check for coercivity. From $(43)_1$ we have, for arbitrary $\boldsymbol{z} = (\boldsymbol{v}, \boldsymbol{Q}, \xi) \in \mathscr{Z}$,

$$\begin{aligned}
a(\boldsymbol{z},\boldsymbol{z}) &= \int_\Omega \mathscr{C}(\boldsymbol{E}(\boldsymbol{v}) - \boldsymbol{Q}) : (\boldsymbol{E}(\boldsymbol{v}) - \boldsymbol{Q})\,\mathrm{d}x + \int_\Omega k_1\xi^2\,\mathrm{d}x + \alpha L^2 \int_\Omega [(1-\beta)|\mathrm{curl}\,\boldsymbol{Q}|^2 \\
&\quad + \beta|\mathrm{skw}\,\mathrm{curl}\,\boldsymbol{Q}|^2]\,\mathrm{d}x \geqslant m_0 \int_\Omega |\boldsymbol{E}(\boldsymbol{v}) - \boldsymbol{Q}|^2\,\mathrm{d}x \\
&\quad + k_1\int_\Omega |\xi|^2\mathrm{d}x \quad (\text{using } (13) \geqslant m_0\left[\int_\Omega |\boldsymbol{E}(\boldsymbol{v})|^2\,\mathrm{d}x - \left(\theta\int_\Omega |\boldsymbol{E}(\boldsymbol{v})|^2\mathrm{d}x + \frac{1}{\theta}\int_\Omega |\boldsymbol{Q}|^2\,\mathrm{d}x\right)\right. \\
&\quad \left. + \int_\Omega |\boldsymbol{Q}|^2\,\mathrm{d}x\right] + k_1\int_\Omega |\xi|^2\,\mathrm{d}x \\
&\quad (\text{using Young's inequality } 2ab \leqslant \theta a^2 + b^2/\theta) \\
&\geqslant m_0(1-\theta)\int_\Omega |\boldsymbol{E}(\boldsymbol{v})|^2\,\mathrm{d}x \\
&\quad + m_0\left(1 - \frac{1}{\theta}\right)\int_\Omega |\boldsymbol{Q}|^2\,\mathrm{d}x + k_1\int_\Omega |\xi|^2\,\mathrm{d}x \geqslant m_0(1-\theta)\int_\Omega |\boldsymbol{E}(\boldsymbol{v})|^2\,\mathrm{d}x + m_0\left(1 - \frac{1}{\theta}\right) \\
&\quad \times \int_\Omega |\boldsymbol{Q}|^2\,\mathrm{d}x + \frac{k_1}{2}\int_\Omega (|\boldsymbol{Q}|^2 + \ell^2|\nabla\boldsymbol{Q}|^2)\,\mathrm{d}x + \frac{k_1}{2}\int_\Omega |\xi|^2\mathrm{d}x \quad (\text{using } (57)) \\
&= m_0(1-\theta)\int_\Omega |\boldsymbol{E}(\boldsymbol{v})|^2\,\mathrm{d}x + \left(m_0\left(1 - \frac{1}{\theta}\right) + k_1\right)\int_\Omega |\boldsymbol{Q}|^2\,\mathrm{d}x + \frac{k_1\ell}{2}\int_\Omega |\nabla\boldsymbol{Q}|^2\,\mathrm{d}x \\
&\quad + \frac{k_1}{2}\int_\Omega \xi^2\,\mathrm{d}x = m_0(1-\theta)\int_\Omega |\boldsymbol{E}(\boldsymbol{v})|^2\,\mathrm{d}x + \frac{k_1\ell}{2}\int_\Omega |\nabla\boldsymbol{Q}|^2\,\mathrm{d}x + \frac{k_1}{2}\int_\Omega \xi^2\,\mathrm{d}x.
\end{aligned} \tag{60}$$

It follows that the bilinear form will be coercive provided that the coefficients of all the integrals are positive, since then we will then have

$$a(\boldsymbol{z},\boldsymbol{z}) \geqslant \alpha\int_\Omega (|\boldsymbol{E}(\boldsymbol{v})|^2 + |\nabla\boldsymbol{Q}|^2 + \xi^2)\,\mathrm{d}x, \tag{61}$$

where $\alpha = \min(m_0(1-\theta), k_1\ell/2, k_1/2)$. Thus by choosing θ such that $0 < \theta < 1$, we see that the bilinear form is coercive, and that therefore a unique solution exists, provided that $k_1 > 0$; that is, provided that we have *hardening behaviour*.

Remark

1. The terms in $(60)_1$ that involve $\mathrm{curl}\,\boldsymbol{Q}$ play no role in establishing the coercivity of $a(\cdot,\cdot)$.

2. The problem is nevertheless *not coercive* for the case of *classical plasticity*, for which the terms involving curls are absent. The definition of $\mathscr{W}$, and therefore the relationship between the hardening parameter and plastic strains and their *gradients*, are crucial to the establishment of coercivity.

4. Softening behaviour

It is seen that the hardening constant k_1 plays a significant role in the proof of coercivity. Indeed, the proof does not hold unless $k_1 > 0$.

The role played by softening in situations in which size effects are relevant has been discussed extensively (see Engelen et al. (2006) for an account that reviews earlier contributions to the subject).

The inability to demonstrate coercivity in softening situations for the model under consideration is in contrast to that for the case of the model studied by Mühlhaus and Aifantis (1991), in which the yield condition contains terms involving not only γ, as is the case here, but also laplacian and biharmonic terms involving γ (see Zbib and Aifantis (1988) for a discussion of the role of higher-order gradients in accommodating softening, and Djoko et al. (in press-a, in press-b) for a detailed analysis of this problem and its numerical approximation). Even when only the laplacian term is retained, one may show that the problem is coercive in the case of softening, provided that the material constant associated with the laplacian term is positive and sufficiently large. This is because we then have

$$\bar{a}(\boldsymbol{z},\boldsymbol{z}) = \int_\Omega \mathscr{C}(\boldsymbol{E}(\boldsymbol{v}) - \boldsymbol{Q}) : (\boldsymbol{E}(\boldsymbol{v}) - \boldsymbol{Q})\,\mathrm{d}x + \int_\Omega k_1 \xi^2\,\mathrm{d}x + \int_\Omega \bar{k}|\nabla\xi|^2\,\mathrm{d}x \tag{62}$$

the last term arising from a yield function that takes the form

$$\overline{\varphi}(\boldsymbol{T}, g) = |\boldsymbol{T}^{\mathrm{D}}| + \overline{g} - c_0$$

this time with $\overline{g}(\gamma) = -k_1\gamma + \bar{k}\nabla^2\gamma$, where $\bar{k} > 0$. Since gradients of the hardening parameter appear in the bilinear form, the space in which these are sought is no longer $\mathscr{M}$ in (52), but rather the space

$$\overline{\mathscr{M}} = \left\{ \xi \,\middle|\, \int_\Omega |\nabla\xi|^2\,\mathrm{d}x \text{ is finite, } \xi = 0 \text{ on the boundary of } \Omega \right\}.$$

That is, we choose $\overline{\mathscr{M}}$ to be the space of hardening parameters with finite 'energy'. Furthermore, it is necessary now to add a boundary condition, which for simplicity we assume to be $\gamma = 0$. More sophisticated boundary conditions such as those for the displacement are easily accommodated. Then $\overline{\mathscr{M}}$ is a Hilbert space with the norm

$$\|\xi\|^2_{\overline{\mathscr{M}}} = \int_\Omega |\nabla\xi|^2\,\mathrm{d}x.$$

The Poincaré inequality (Evans, 1998), which states that there exists a constant $c_P > 0$ such that

$$c_P \int_\Omega \xi^2\,\mathrm{d}x \leqslant \int_\Omega |\nabla\xi|^2\,\mathrm{d}x \tag{63}$$

for any function with the property that $\xi = 0$ on at least a part of the boundary, ensures that $\|\cdot\|_{\overline{\mathscr{M}}}$ is a norm on $\mathscr{M}$. The space of admissible displacements is unchanged, while the space of admissible plastic strains is now

$$\overline{\mathscr{Q}} = \left\{ \boldsymbol{Q} | \int_\Omega |\boldsymbol{Q}|^2 \, \mathrm{d}x \text{ is finite} \right\}$$

with norm

$$\|\boldsymbol{Q}\|^2_{\overline{\mathscr{Q}}} = \int_\Omega \|\boldsymbol{Q}\|^2 \, \mathrm{d}x$$

since no derivatives of the plastic strain are present.

After making the necessary changes to the definition of $\mathscr{Z}$ the set of admissible solutions is now

$$\overline{\mathscr{W}} = \{\boldsymbol{z} \in \mathscr{Z} \mid |\boldsymbol{Q}| \leqslant \xi\}. \tag{64}$$

Proceeding exactly as in (60), and considering the case in which $k_1 < 0$, we have

$$a(\boldsymbol{z},\boldsymbol{z}) \geqslant m_0(1-\theta)\int_\Omega |\boldsymbol{E}(\boldsymbol{v})|^2 \, \mathrm{d}x + m_0\left(1-\frac{1}{\theta}\right)\int_\Omega |\boldsymbol{Q}|^2 \, \mathrm{d}x + k_1 \int_\Omega |\xi|^2 \, \mathrm{d}x + \bar{k}\int_\Omega |\nabla \xi|^2 \, \mathrm{d}x. \tag{65}$$

By making use of (63) and the definition of $\overline{\mathscr{W}}$ we obtain

$$a(\boldsymbol{z},\boldsymbol{z}) \geqslant m_0(1-\theta)\int_\Omega |\boldsymbol{E}(\boldsymbol{v})|^2 \, \mathrm{d}x + \left(m_0\left(1-\frac{1}{\theta}\right) + \frac{\bar{k}}{4}c_P\right)\int_\Omega |\boldsymbol{Q}|^2 \, \mathrm{d}x + \left(k_1 + \frac{\bar{k}}{4}c_P\right) \times \int_\Omega |\xi|^2 \, \mathrm{d}x + + \frac{\bar{k}}{2}\int_\Omega |\nabla \xi|^2 \, \mathrm{d}x. \tag{66}$$

It follows that the bilinear form is coercive provided that:

$$k_1 > -\frac{\bar{k}c_P}{4} \quad \text{and} \quad 0 < \theta < \frac{4m_0}{4m_0 + \bar{k}c_P} < 1. \tag{67}$$

Thus the model is able to accommodate a degree of softening which, unsurprisingly, depends on the magnitude of the parameter associated with the gradient term.

Since a similar analysis is not possible for the model studied in Section 3, this prompts the question as to the conditions that need to be added to that model in order to achieve well-posedness in the softening range. In other words, how does one compensate for a situation in which $k_1 \leqslant 0$?

We explore one avenue, prompted by the mathematical structure of the problem rather than by physical considerations, which is to require that the free energy be a function of both the curl and the divergence of the plastic strain. This leads to a modification of (10) so that it now reads

$$\psi = \widetilde{\psi}(\boldsymbol{E}^{\mathrm{e}}, \boldsymbol{G}, \boldsymbol{g}, \gamma) \tag{68}$$

in which $\boldsymbol{g} = \operatorname{div} \boldsymbol{E}^{\mathrm{p}}$. We assume a quadratic dependence of the free energy on $\boldsymbol{g}$, so that

$$\widetilde{\psi}(\boldsymbol{E}^{\mathrm{e}}, \boldsymbol{G}, \boldsymbol{g}, \gamma) = \hat{\psi}(\boldsymbol{E}^{\mathrm{e}}, \boldsymbol{G}, \gamma) + \frac{1}{2}k_3|\boldsymbol{g}|^2 \quad \text{with } k_3 > 0 \tag{69}$$

with $\hat{\psi}$ given by (11). With the free-energy imbalance still given by (14), substitution in this inequality of the expression (68) for ψ leads to the dissipation inequality

$$\boldsymbol{T}^{\mathrm{p}} : \dot{\boldsymbol{E}}^{\mathrm{p}} + \mathbb{K}^{\mathrm{p}}_{\mathrm{dis}} \,\vdots\, \nabla\dot{\boldsymbol{E}}^{\mathrm{p}} + \boldsymbol{h}\cdot\dot{\boldsymbol{g}} + g\dot{\gamma} \geqslant 0, \tag{70}$$

in which

$$\boldsymbol{h} := -\frac{\partial\tilde{\psi}}{\partial\boldsymbol{g}} = -k_3\boldsymbol{g}.$$

We may write

$$\boldsymbol{h}\cdot\dot{\boldsymbol{g}} = h_i\dot{E}^{\mathrm{p}}_{ij,j} = H_{ijk}\dot{E}^{\mathrm{p}}_{ij,k} = \mathbb{H} \,\vdots\, \nabla\dot{\boldsymbol{E}}^{\mathrm{p}}, \tag{71}$$

where $\mathbb{H}$ is the third-order tensor defined by

$$H_{ijj} = h_i, \ \ (\text{no sum on } j), \ \text{all other components zero.} \tag{72}$$

Then (70) becomes

$$\boldsymbol{T}^{\mathrm{p}} : \dot{\boldsymbol{E}}^{\mathrm{p}} + \widetilde{\mathbb{K}}^{\mathrm{p}}_{\mathrm{dis}} \,\vdots\, \nabla\dot{\boldsymbol{E}}^{\mathrm{p}} + g\dot{\gamma} \geqslant 0, \tag{73}$$

with

$$\widetilde{\mathbb{K}}^{\mathrm{p}}_{\mathrm{dis}} = \mathbb{K}^{\mathrm{p}}_{\mathrm{dis}} + \mathbb{H}. \tag{74}$$

The only changes that need be made in the flow law entail the replacement of $\boldsymbol{\Sigma}$ by $\widetilde{\boldsymbol{\Sigma}} := (\boldsymbol{T}^{\mathrm{p}}, \widetilde{\mathbb{K}}^{\mathrm{p}}_{\mathrm{dis}}, g)$ in (19) and in every other subsequent occurrence. The end result of this modification is the replacement of (41) by

$$\begin{aligned}\int_\Omega D(\boldsymbol{Q},\nabla\boldsymbol{Q},\xi)\,\mathrm{d}x \geqslant{}& \int_\Omega D(\dot{\boldsymbol{E}}^{\mathrm{p}},\nabla\dot{\boldsymbol{E}}^{\mathrm{p}},\dot{\gamma})\,\mathrm{d}x + \int_\Omega \boldsymbol{T} : (\boldsymbol{Q}-\dot{\boldsymbol{E}}^{\mathrm{p}})\,\mathrm{d}x \\ &- \int_\Omega \frac{\partial\hat{\psi}}{\partial\boldsymbol{G}} : \mathrm{curl}\,(\boldsymbol{Q}-\dot{\boldsymbol{E}}^{\mathrm{p}})\,\mathrm{d}x - \int_\Omega k_3\,\mathrm{div}\,\boldsymbol{E}^{\mathrm{p}}\cdot\mathrm{div}\,(\boldsymbol{Q}-\dot{\boldsymbol{E}}^{\mathrm{p}})\,\mathrm{d}x + \int_\Omega g(\xi-\dot{\gamma})\,\mathrm{d}x.\end{aligned} \tag{75}$$

The bilinear form $a(\cdot,\cdot)$ is now replaced by $\tilde{a}(\cdot,\cdot)$, where

$$\tilde{a}(\boldsymbol{w},\boldsymbol{z}) = a(\boldsymbol{w},\boldsymbol{z}) + \int_\Omega k_3\,\mathrm{div}\,\boldsymbol{E}^{\mathrm{p}}\cdot\mathrm{div}\,\boldsymbol{Q}\,\mathrm{d}x \tag{76}$$

in full, this is

$$\begin{aligned}\tilde{a}(\boldsymbol{w},\boldsymbol{z}) ={}& \int_\Omega \mathscr{C}(\boldsymbol{E}(\boldsymbol{u})-\boldsymbol{E}^{\mathrm{p}}) : (\boldsymbol{E}(\boldsymbol{v})-\boldsymbol{Q})\,\mathrm{d}x + \int_\Omega k(\gamma)\xi\,\mathrm{d}x \\ &+ \int_\Omega \mu L^2[\beta\,\mathrm{s}\,\mathrm{curl}\,\boldsymbol{E}^{\mathrm{p}} : \mathrm{curl}\,\boldsymbol{Q}\,\mathrm{d}x + 4(1-\beta)(\mathrm{skw}\,\mathrm{curl}\,\boldsymbol{E}^{\mathrm{p}}) : (\mathrm{skw}\,\mathrm{curl}\,\boldsymbol{Q})]\,\mathrm{d}x \\ &+ \int_\Omega k_3\,\mathrm{div}\,\boldsymbol{E}^{\mathrm{p}}\cdot\mathrm{div}\,\boldsymbol{Q}\,\mathrm{d}x.\end{aligned} \tag{77}$$

The spaces on which the problem is posed are unchanged, since the Friedrichs inequality states that for functions in $\mathscr{Q}$ there exists a constant $C_{\mathrm{F}} > 0$ such that

$$C_{\mathrm{F}}\int_\Omega |\nabla\boldsymbol{Q}|^2\,\mathrm{d}x \leqslant \int_\Omega \left(|\mathrm{div}\,\boldsymbol{Q}|^2 + |\mathrm{curl}\,\boldsymbol{Q}|^2\right)\mathrm{d}x \tag{78}$$

provided that Ω is simply-connected (see Girault and Raviart, 1986). Thus div $\boldsymbol{Q}$ may be bounded above and below by $\nabla \boldsymbol{Q}$.

It turns out that it is still not possible to show coercivity on all of $\mathscr{W}$ (see (57), but only on a subset on which we have $|\boldsymbol{Q}| = \xi$. For this case, and assuming as before linear softening behaviour with $k_1 \leqslant 0$, retracing the steps in (60), and using (78), we have

$$\begin{aligned}
\tilde{a}(\boldsymbol{z},\boldsymbol{z}) &\geqslant \int_\Omega m_0|\boldsymbol{E}(\boldsymbol{v}) - \boldsymbol{Q}|^2 \mathrm{d}x + \int_\Omega k_1|\xi|^2 \mathrm{d}x + \int_\Omega \mu L^2 \beta |\mathrm{curl}\,\boldsymbol{Q}|^2 \mathrm{d}x + \int_\Omega k_3 |\mathrm{div}\,\boldsymbol{Q}|^2 \mathrm{d}x \\
&\geqslant m_0(1-\theta)\int_\Omega |\boldsymbol{E}(\boldsymbol{v})|^2 \mathrm{d}x + m_0\left(1-\frac{1}{\theta}\right)\int_\Omega |\boldsymbol{Q}|^2 \mathrm{d}x + k_1 \int_\Omega \xi^2 \mathrm{d}x \\
&\quad + \underbrace{C_\mathrm{F}\min(\mu L^2\beta, k_3)}_{C_Q}\int_\Omega |\nabla \boldsymbol{Q}\|^2 \mathrm{d}x \geqslant m_0(1-\theta)\int_\Omega |\boldsymbol{E}(\boldsymbol{v})|^2 \mathrm{d}x + \left[m_0\left(1-\frac{1}{\theta}\right) + \frac{C_Q}{2}\right] \\
&\quad \times \int_\Omega |\boldsymbol{Q}|^2 \mathrm{d}x + k_1 \int_\Omega \xi^2 \mathrm{d}x + \underbrace{\frac{C_Q}{2}\min(1/\ell^2, 1)}_{C_W}\int_\Omega \left(|\boldsymbol{Q}|^2 + \ell^2 |\nabla \boldsymbol{Q}\|^2\right)\mathrm{d}x \geqslant m_0(1-\theta) \\
&\quad \times \int_\Omega |\boldsymbol{E}(\boldsymbol{v})|^2 \mathrm{d}x + \left[m_0\left(1-\frac{1}{\theta}\right) + \frac{C_Q}{2}\right]\int_\Omega |\boldsymbol{Q}|^2 \mathrm{d}x + \left(k_1 + \frac{C_W}{2}\right)\int_\Omega \xi^2 \mathrm{d}x \\
&\quad + \frac{C_W}{2}\int_\Omega (|\boldsymbol{Q}|^2 + \ell^2 |\nabla \boldsymbol{Q}\|^2)\mathrm{d}x \geqslant C(\|\boldsymbol{v}\|^2_{\mathscr{V}} + \|\boldsymbol{Q}\|^2_{\mathscr{Q}} + \|\xi\|^2_{\mathscr{M}}).
\end{aligned} \tag{79}$$

By choosing θ such that $1 > \theta > 2m_0/(2m_0 + c_Q)$ it is seen that the bilinear form is coercive provided that the softening is limited by $|k_1| < C_W/2$.

The result (79) unfortunately is valid not on all of $\mathscr{W}$, but only on the nonconvex subset on which the plastic strains and hardening parameters satisfy $|\boldsymbol{Q}|^2 + \ell^2|\nabla \boldsymbol{Q}|^2 = \xi^2$. This is not sufficient to guarantee coercivity, and hence well-posedness, so that a rigorous result on well-posedness for the case of softening is still elusive. The result (79) does however indicate that solutions corresponding to the model with the modified free energy, and for which (26) remains valid, do in fact render the problem coercive. A more general result on well-posedness for the case of softening would, however, appear to require the inclusion of higher-order terms involving the hardening parameter γ, in the spirit of Mühlhaus and Aifantis (1991).

5. Concluding remarks

The initial boundary value problem corresponding to the model of strain gradient plasticity proposed in Gurtin and Anand (2005a) has been formulated as a variational inequality, and its well-posedness explored. For situations in which hardening occurs the model is well-posed. When softening is present it is not possible to show coercivity; an extension of the model in which the divergence of the plastic strain is included as a variable renders the problem coercive, but on a nonconvex subset, which does not allow the general theorem on well-posedness to be applied. Since the solution, if it exists, satisfies $\sqrt{|\dot{\boldsymbol{E}}^\mathrm{p}|^2 + \ell^2|\nabla \dot{\boldsymbol{E}}|^2} = \dot{\gamma}$ (see (26)), it is seen that the triple $(\dot{\boldsymbol{E}}^\mathrm{p}, \nabla \dot{\boldsymbol{E}}^\mathrm{p}, \dot{\gamma})$ does lie in this subset. This does not suffice to conclude existence but it does demonstrate, in an informal way, that softening solutions may be accommodated by the extended model. It has been shown that the model proposed in Mühlhaus and Aifantis (1991) does not have the same drawbacks, and that it accommodates at least a certain degree of softening.

The analysis presented here demonstrates the value of qualitative analyses of problems of this nature, in that the results obtained in this way shed light on properties of the problem that might not otherwise be evident.

Corresponding analyses of problems associated with other models of strain gradient plasticity await a similar treatment. A key goal is that of investigating the properties of those models that include in their formulation higher-order derivatives of plastic strain, and indeed in some instances of the displacement as well. Examples include the model of Fleck and Hutchinson (1997), in which second derivatives of displacement and of plastic strain would be present in the weak or variational form, the 'full' model of Mühlhaus and Aifantis (1991) which includes a term involving the biharmonic of the hardening parameter, and the model of Menzel and Steinmann (2000), for which case a particular combination of second derivatives, that is, the quantity $\mathrm{curl\,curl}\,\boldsymbol{E}^{\mathrm{p}}$ would be required to be square-integrable, and which represents a particularly interesting set of mathematical challenges. Finally, a further valuable extension would be an analysis of the situation in which more than one length scale is present. A primary example of such a theory is that of Fleck and Hutchinson (2001), which extends the theory of Aifantis to one involving three length scales. All of these extensions are the subject of current investigation.

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