Local and Global Existence of Solutions to the Equations for Fibre Suspension Flows [1]

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Abstract

We establish the existence and uniqueness, locally and globally in time, of solutions to the governing equations for fibre suspensions flows, for sufficiently small data. The linear and quadratic closure rules are considered, and the rotary diffusivity is assumed to be constant. The existence of a unique classical solution, local in time, is proven by using the Schauder Fixed Point Theorem, for both the linear and quadratic closures. Global a priori estimates are then derived to obtain a unique global classical solution for sufficiently small data. The solution is found to be stable in the absence of a body force. Existence and uniqueness of solutions to the steady problem are also established.

Keywords: fibre suspensions, global existence, stability

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1 Introduction

Mathematical models of fibre suspension flows take the form of a coupled system of nonlinear partial differential equations, which bear some resemblance to the equations governing the flow of viscoelastic liquids (see, for example, [9]). The primary variables are velocity and pressure, as in the case of Newtonian fluids, and in addition a tensorial variable known as the orientation tensor, which captures in an average sense the orientation of fibres in space and time.

The orientation tensor is one of second-order. The governing equations contain also a fourth-order orientation tensor, which is normally expressed as a function of that of second order through a closure approximation. Many such approximations exist [1, 2, 3, 4, 17], and it is known that while particular closure approximations represent accurate models in certain flow situations, there is no known universally appropriate closure rule.

There has recently been a series of theoretical studies aimed at gaining a better understanding of the properties of the equations governing the flow of fibre suspensions. In a study of the well-posedness of these equations, Galdi and Reddy [8] have shown that for the linear closure there is a connection between stability and the particle number, a constant closely related to fibre concentration: in particular, the rest state is unstable, in the sense of Liapounov, for particle numbers exceeding 35/2. These authors also demonstrate the existence, locally in time, of a solution to the equations corresponding to the quadratic closure, and for the case in which the rotary diffusivity is proportional to $|\boldsymbol{D}|$, the magnitude of the stretching tensor; this approximation is due to Folgar and Tucker [6].

Munganga et al. [12] have studied the stability of fibre suspension flows from thermodynamic and energetic perspectives, for a wide range of closure approximations. Their results for the linear closure confirm those obtained in [8], in that a necessary condition for satisfaction of the dissipation inequality, and a sufficient condition for energetic stability, are that the particle number be less than 35/2.

The aim of this work is to establish conditions for the existence of a unique classical solution, locally and globally in time, to the equations for fibre suspension flows, for the case in which the linear and quadratic closure are used, and for a constant value of the rotary diffusivity D_r. As indicated earlier, local existence of solutions was established for the case in which D_r satisfies the Folgar-Tucker assumption; in that case it was not possible to establish global existence of solutions. The case of constant rotary diffusivity is one of interest, since such a model is especially appropriate for suspensions of small fibres, in which rotary Brownian motion may be present [16].

The methods used here are based on those employed by GUILLOPÉ AND SAUT [9] in their study of the equations for viscoelastic fluids. We also make use of techniques employed in GALDI AND REDDY [8].

The proof of existence of a local solution uses a fixed point argument. Global a priori estimates are derived to obtain a unique global classical solution for sufficiently small data, and it is proven that that solution is stable in the absence of body forces. We also derive results on the existence and uniqueness of steady flows.

The rest of this paper is organised as follows. In Section 2 the mechanics of fibre suspension flows is reviewed. The problem of local existence of solutions is investigated in Section 3, for both linear and quadratic closure rules, while the global existence of solutions and the stability of those solutions are the subjects of Section 4. In Section 5 the existence and uniqueness of solutions to the steady problem are established.

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Notation and function spaces. We make use of coordinate-free notation wherever convenient, and denote vectors and tensors by bold-face letters.

The scalar product of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is denoted by $\boldsymbol{u} \cdot \boldsymbol{v}$, while the corresponding product of two second-order tensors $\boldsymbol{A}$ and $\boldsymbol{B}$ is denoted by $\boldsymbol{A}{:}\boldsymbol{B}$. In index form these expressions read $u_i v_i$ and $A_{ij}B_{ij}$, the summation convention on repeated indices being applied at all times. The magnitude of a vector $\boldsymbol{u}$ and a tensor $\boldsymbol{A}$ are then naturally defined by $|\boldsymbol{u}| = (\boldsymbol{u} \cdot \boldsymbol{u})^{1/2}$ and $|\boldsymbol{A}| = (\boldsymbol{A} : \boldsymbol{A})^{1/2}$.

In what follows, Ω will denote a bounded domain in $\mathbb{R}^d$ ($d = 2$ or 3), with boundary Γ.

We will assume that Ω is on one side of Γ, and that Γ is at least Lipschitzian or C^1. We set $\Omega_T = \Omega \times T$, for $T > 0$.

We denote by $C(\Omega)$ the space of all real-valued continuous functions on Ω, and by $C(\overline{\Omega})$ the space of functions that are *bounded and uniformly continuous* on Ω. The space $C(\overline{\Omega})$ is a *Banach space* with the norm

$$\|v\|_{C(\overline{\Omega})} = \sup\{|v(\boldsymbol{x})| : \ \boldsymbol{x} \in \Omega\} \equiv \max\{|v(\boldsymbol{x})| : \ \boldsymbol{x} \in \overline{\Omega}\}.$$

For any nonnegative integer m, we likewise set

$$C^m(\overline{\Omega}) = \{v \in C(\overline{\Omega}) : \ D^\alpha v \in C(\overline{\Omega}) \text{ for } |\alpha| \le m\}.$$

$C(\Omega)$ and $C(\overline{\Omega})$ will denote $C^0(\Omega)$ and $C^0(\overline{\Omega})$ respectively. The space $C^m(\overline{\Omega})$ is a Banach space when endowed with the norm

$$\|v\|_{C^m(\overline{\Omega})} = \sum_{|\alpha| \le m} \|D^\alpha v\|_{C(\overline{\Omega})}.$$

We define

$$C^\infty(\overline{\Omega}) = \{v \in C(\overline{\Omega}) : \ v \in C^m(\overline{\Omega}) \ \forall \, m \in \mathbb{Z}_+\}.$$

We denote by $L^p(\Omega)$ $(p \ge 1)$ the Lebesgue spaces consisting of equivalence classes of functions whose pth powers are integrable on Ω; these are Banach spaces with the norm

$$\|v\|_{L^p(\Omega)} = \left[\int_\Omega |v(\boldsymbol{x})|^p \ dx\right]^{1/p}.$$

The modification for $p = \infty$ is made in the usual way.

The Sobolev spaces $H^k(\Omega)$, $k = 0, 1, \ldots$ are defined by

$$H^k(\Omega) = \{v \in L^2(\Omega), \ D^\alpha v \in L^2(\Omega), \ |\alpha| \le k\},$$

where $\alpha = (\alpha_1, \cdots, \alpha_d)$, α_i being nonnegative integers. These are Hilbert spaces when endowed with the inner product

$$(u, v)_{H^k} = \sum_{|\alpha| \le k} \int_\Omega D^\alpha u D^\alpha v \ dx.$$

The space $H_0^k(\Omega)$ is defined by

$$H_0^k(\Omega) = \{v \in H^k(\Omega): \ v|_\Gamma = 0 \ \text{ and } D^\alpha f = 0 \text{ on } \Gamma \ \text{ for } |\alpha| < k\}.$$

The seminorm $\|v\| := |\nabla v|$ is a norm on $H_0^1(\Omega)$, equivalent to the natural norm.

The topological dual of $H_0^k(\Omega)$ is denoted by $H^{-k}(\Omega)$.

We will require spaces of vector- or tensor-valued functions whose components are members of a Sobolev space. Thus we denote by $\mathbb{L}^p(\Omega)$ the space of vector- or tensor-valued functions with components in $L^p(\Omega)$. The space $\mathbb{H}^k(\Omega)$ is defined likewise. We will also require the spaces

$$\begin{aligned}
&\mathbb{H} = \{\boldsymbol{v} : v_i \in L^2(\Omega), \ \text{div}\ \boldsymbol{v} = 0 \ \text{ in } \ \Omega, \ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \ \text{ on } \ \ \Gamma\},\\
&\mathbb{V} = \{\boldsymbol{v} : \ v_i \in H_0^1(\Omega), \ \text{div}\ \boldsymbol{v} = 0 \ \text{ in } \ \Omega\},\\
&\mathbb{X} = \{\boldsymbol{A} : \ A_{ij} \in L^2(\Omega), \ A_{ij} = A_{ji}, \ A_{ii} \ = \ 0 \quad \text{a.e in } \ \Omega\}.
\end{aligned}$$

These spaces are equipped respectively with the norms

$$\begin{aligned}
&\| \cdot \|_{\mathbb{H}} \equiv \| \cdot \|_{\mathbb{L}^2} \equiv | \cdot |;\\
&\| \cdot \|_{\mathbb{V}} \equiv \| \cdot \|_{\mathbb{H}^1} \qquad \text{and}\\
&\| \cdot \|_{\mathbb{X}} \equiv \| \cdot \|_{\mathbb{L}^2}.
\end{aligned}$$

We denote by P the orthogonal projection of $\mathbb{L}^2(\Omega)$ onto $\mathbb{H}$, and the operator $\mathcal{L}$ by

$$\mathcal{L}(\boldsymbol{v}) = -P\Delta\boldsymbol{v}, \tag{1.1}$$

where Δ is the Laplacian operator. The domain of $\mathcal{L}$ is given by $D(\mathcal{L}) = \mathbb{V}\cap\mathbb{H}^2(\Omega)$, and the norm $\|\boldsymbol{v}\|_{D(\mathcal{L})} := \|\mathcal{L}\boldsymbol{v}\|_{L^2}$ is equivalent to the natural $\mathbb{H}^2$-norm.

Finally, we will require various Bochner spaces; these are spaces of functions which are maps from a time interval to a Banach or Hilbert space. Let X be a Banach space and T a positive number; then the space $C^m(0,T;X)$ consists of all m-times continuously differentiable functions v from $[0,T]$ to X. This is a Banach space with the norm

$$\|v\|_{C^m(0,T;X)} = \sum_{k=0}^{m} \max_{0\le t\le T} \|v^{(k)}(t)\|_X.$$

For $1 \leq p < \infty$ the space $L^p(0,T;X)$ consists of all measurable functions v from $(0,T)$ to X for which

$$\|v\|_{L^p(0,T;X)} := \left(\int_0^T \|v(t)\|_X^p \, dt \right)^{1/p} < \infty.$$

This is a Banach space with the norm $\|v\|_{L^p(0,T;X)}$. The extension to the case $p = \infty$ is carried out in the usual way, as is the extension to vector- or tensor-valued functions.

2 Mechanics of fibre suspension flows

We give a brief but self-contained review of the equations governing the flow of fibre suspensions. Detailed accounts may be found, for example, in the works of Advani and Tucker [1, 2, 15].

Fibre suspensions. For the purposes of this investigation, a fibre suspension is defined to be a viscous incompressible fluid in which is suspended a distribution of axisymmetric, rigid, and ideally massless fibres. The fibre volume fraction is denoted by h, and the fibre aspect ratio r is defined by $r = \ell/d$, where ℓ and d are respectively the fibre length and diameter. The orientation of each fibre is described by a unit vector $\boldsymbol{p}$.

A suspension is said to be

$$\left.\begin{array}{r} \text{dilute} \\ \text{semi-dilute} \\ \text{concentrated} \end{array}\right\} \text{ if } \left\{\begin{array}{l} hr^2 < 1 \\ 1 < hr^2 < r \\ r < hr^2 \end{array}\right. . \tag{2.1}$$

We consider dilute and semi-dilute suspensions, for which fibres have a low probability of making contact, though the motion of the fibres and fluid are coupled.

The suspension occupies a bounded domain Ω in $\mathbb{R}^d$ $(d = 2, 3)$.

A continuum theory for fibre suspensions may be constructed by introducing the probability density function $\psi(\boldsymbol{p})$, which gives the probability of a fibre having that particular orientation. The probability density function is not a convenient variable with which to work, but it may be eliminated in favour of orientation tensors, which are field variables describing, in an averaged sense, the distribution of orientation in the domain Ω.

We define, for any function f, the averaging operator , $< \cdot >$ by

$$< f > = \int f(\theta, \phi)\psi(\theta, \phi)\sin\theta d\theta d\phi, \tag{2.2}$$

in which integration is over the unit sphere. Then the second- and fourth-order orientation tensors $\boldsymbol{A}$ and $\mathcal{A}$ are defined by

$$\begin{aligned} \boldsymbol{A} &= \langle \boldsymbol{p} \otimes \boldsymbol{p} \rangle, && (2.3) \\ \mathcal{A} &= \langle \boldsymbol{p} \otimes \boldsymbol{p} \otimes \boldsymbol{p} \otimes \boldsymbol{p} \rangle. && (2.4) \end{aligned}$$

It is easily shown that orientation tensors of odd order are identically zero.

By making use of these definitions and of the equations of conservation of fibres, and of the equation of motion of a single particle, it is possible to derive an evolution equation for the tensor $\boldsymbol{A}$: this is given by [17]

$$\frac{D\boldsymbol{A}}{Dt} = (\boldsymbol{W}\boldsymbol{A} - \boldsymbol{A}\boldsymbol{W}) + \lambda(\boldsymbol{D}\boldsymbol{A} + \boldsymbol{A}\boldsymbol{D} - 2\mathcal{A}\boldsymbol{D}) + D_r(\boldsymbol{I} - d\boldsymbol{A}). \tag{2.5}$$

Here $\dfrac{D\boldsymbol{A}}{Dt} \equiv \dfrac{\partial \boldsymbol{A}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{A}$ is the material derivative, $\boldsymbol{D} = \frac{1}{2}[\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^T]$ is the stretching tensor, and $\boldsymbol{W} = \frac{1}{2}[\nabla\boldsymbol{v} - (\nabla\boldsymbol{v})^T]$ is the spin tensor, corresponding to a velocity field $\boldsymbol{v}$; the quantity $\lambda = \dfrac{r^2 - 1}{r^2 + 1}$ is a parameter that characterises particle slenderness.

The parameter D_r represents the rotary diffusivity. The case $D_r =$ constant is an excellent model for very small particles which experience rotary Brownian motion [16], and this assumption is adopted throughout this work.

Closure approximations. Equation (2.5) contains the tensor $\mathcal{A}$; in fact it is a feature of such evolution equations that the equation for an orientation tensor of a particular rank contains the tensor of the next (even) rank up. In order to circumvent this problem, $\mathcal{A}$ is conventionally approximated by means of what is known as a closure rule. This approximation consists of writing $\mathcal{A}$ as a function of the second-order tensor $\boldsymbol{A}$. There has been a great deal of work on closures (see for example ADVANI AND TUCKER [1], HINCH AND LEAL [10], CINTRA AND TUCKER [3] and DUPRET AND VERLEYE [4]); we focus on two examples, viz. the *linear* and *quadratic* approximations. The linear approximation $\mathcal{A}^L$ is defined by

$$\mathcal{A}^L\boldsymbol{D} = -\tfrac{2}{35}\boldsymbol{D} + \tfrac{1}{7}[2\boldsymbol{A}\boldsymbol{D} + 2\boldsymbol{D}\boldsymbol{A} + (\boldsymbol{A} : \boldsymbol{D})\boldsymbol{I}] \tag{2.6}$$

for any symmetric second-order tensor $\boldsymbol{D}$ with $\operatorname{tr}\boldsymbol{D} = 0$, while the quadratic approximation $\mathcal{A}^Q$ is defined by

$$\mathcal{A}^Q = \boldsymbol{A} \otimes \boldsymbol{A}. \tag{2.7}$$

The linear closure rule is exact for random distributions of fibres, while the quadratic closure is exact for perfectly aligned fibres. Each of these approximations is suitable for only a range of physical situations, and there are situations in which neither is a good approximation [2, 8]. Nevertheless, from a mathematical point of view the results on well-posedness that are to be established here may be extended, with little difficulty, to more elaborate closure rules, and so we concentrate on just the two above examples.

Constitutive equation for the stress. Coupling of the fluid and fibre motions implies that the usual constitutive equation for the stress will be modified by terms that contain the orientation tensor. This constitutive equation is given by

$$\boldsymbol{T} = -p\boldsymbol{I} + \boldsymbol{S}, \tag{2.8}$$

where

$$\boldsymbol{S} = 2\mu_I[\boldsymbol{D} + N_p\mathcal{A}\boldsymbol{D} + N_s(\boldsymbol{AD} + \boldsymbol{DA})]. \tag{2.9}$$

Here p is the pressure, μ_I contains all the isotropic contributions to viscosity, while the anisotropic contributions due to the particles are represented by N_p, the *particle number* and N_s, the *shear number.*

The initial boundary value problem. We assume that the fluid occupies a bounded domain $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) with boundary Γ, and is subjected to the action of a body force $\boldsymbol{b}$ per unit mass. The fluid has mass density ρ. It is required to find the velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$, the pressure $p(\boldsymbol{x}, t)$, and the orientation tensor field $\boldsymbol{A}(\boldsymbol{x}, t)$ which satisfy the following set of equations:

conservation of momentum

$$\rho\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} - \operatorname{div}\boldsymbol{T} = \rho\boldsymbol{b}; \tag{2.10}$$

conservation of mass (continuity equation)

$$\operatorname{div}\boldsymbol{v} = 0; \tag{2.11}$$

constitutive equation for stress

$$\boldsymbol{T} = -p\boldsymbol{I} + 2\mu_I[\boldsymbol{D} + N_p\mathcal{A}\boldsymbol{D} + N_S(\boldsymbol{A}\boldsymbol{D} + \boldsymbol{D}\boldsymbol{A})]; \tag{2.12}$$

and the evolution equation for the orientation tensor

$$\frac{D\boldsymbol{A}}{Dt} = (\boldsymbol{W}\boldsymbol{A} - \boldsymbol{A}\boldsymbol{W}) + \lambda(\boldsymbol{D}\boldsymbol{A} + \boldsymbol{A}\boldsymbol{D} - 2\mathcal{A}\boldsymbol{D}) + D_r(\boldsymbol{I} - d\boldsymbol{A}). \tag{2.13}$$

These equations are supplemented by a boundary condition for the velocity field, and initial conditions. For convenience we make use of the homogeneous Dirichlet boundary condition

$$\boldsymbol{v} = \boldsymbol{0} \quad \text{on} \quad \Gamma \tag{2.14}$$

and the inital conditions

$$\boldsymbol{v}(\boldsymbol{x}, 0) = \boldsymbol{v}_0, \quad \boldsymbol{A}(\boldsymbol{x}, 0) = \boldsymbol{A}_0. \tag{2.15}$$

In (2.12) and (2.13) it is assumed that one of the closure approximations will be used to express $\mathcal{A}$ in terms of $\boldsymbol{A}$.

3 Local existence of solutions

In this Section we show, for the cases of the linear and quadratic closures, that the problem (2.10)–(2.15) admits a unique solution locally in time, for the case of constant D_r. GALDI AND REDDY [8] have established the existence of a unique solution locally in time for the case which which uses the Folgar-Tucker assumption $D_r = C|\boldsymbol{D}|$, and for the quadratic closure.

3.1 Linear closure approximation

The dimensionless and traceless problem. We set

$$\begin{gathered}\overline{\boldsymbol{x}} = \frac{\boldsymbol{x}}{L}, \quad \overline{\boldsymbol{v}} = \frac{\boldsymbol{v}}{V} \quad \overline{t} = \frac{tV}{L}, \quad \overline{p} = \frac{pL}{\mu_I V}, \\ \overline{\boldsymbol{b}} = \rho\frac{\boldsymbol{b}L^2}{\mu_I V}, \quad Re = \frac{\rho V L}{\mu_I}, \quad We = \frac{\lambda V}{dD_r L}.\end{gathered} \tag{3.1}$$

Here V and L represent a typical velocity and length of the flow. The constant Re is the Reynolds number, while We is a dimensionless constant that plays a role very similar to that of the Weissenberg number in viscoelastic flows (see, for example, [9]), and for convenience is denoted in the same way.

It is also useful to carry out an additive decompostion of $\boldsymbol{A}$: we set

$$\boldsymbol{A} = \hat{\boldsymbol{A}} + \boldsymbol{A}^* \tag{3.2}$$

where $\boldsymbol{A}^*$ is a diagonal tensor with trace 1, which implies that $\hat{\boldsymbol{A}}$ is traceless. For convenience, and without any loss in generality, we choose

$$\boldsymbol{A}^* = \frac{1}{d}\boldsymbol{I} \tag{3.3}$$

for a problem in $\mathbb{R}^d$. We next make use of (3.1)–(3.3) in equation (2.10) and (2.13), and use (2.6) for $\mathcal{A}$, to obtain

$$\left.\begin{aligned} Re(\boldsymbol{v}' + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}) + \nabla p - \gamma\Delta\boldsymbol{v} &= \boldsymbol{b} + \mathrm{div}\boldsymbol{S}_L \\ \\ \boldsymbol{S}_L = N(\boldsymbol{AD}+\boldsymbol{DA}) &+ \bar{N}_p(\boldsymbol{A}:\boldsymbol{D})\boldsymbol{I} \end{aligned}\right\} \tag{3.4}$$

and

$$\begin{aligned} \boldsymbol{A} + We\left\{\boldsymbol{A}' + (\boldsymbol{v}\cdot\nabla)\boldsymbol{A} + \boldsymbol{AW} - \boldsymbol{WA} - 3\bar{\lambda}(\boldsymbol{AD}+\boldsymbol{DA})\right\} \\ = 2\omega\boldsymbol{D} + 2\bar{\lambda}We(\boldsymbol{A}:\boldsymbol{D})\boldsymbol{I}. \end{aligned} \tag{3.5}$$

For convenience we have denoted $\hat{\boldsymbol{A}}$ by $\boldsymbol{A}$, and we have dispensed also with all overbars. Here

$$\gamma = 2\left(1 - \frac{2}{35}N_p\right), \quad \bar{N}_p = \tfrac{2}{7}N_p, \quad N = \tfrac{2}{7}(2N_p + 7N_s), \quad \bar{\lambda} = \frac{\lambda}{7},$$

and

$$\omega = \frac{13\lambda}{35dD_r}.$$

We introduce the bilinear mapping

$$b(\boldsymbol{v},\boldsymbol{w}) = P(\boldsymbol{v}\cdot\nabla)\boldsymbol{w}. \tag{3.6}$$

In the next section, we implement a fixed point argument, using Schauder's Fixed Point Theorem, to show the existence of a regular solution on a time interval $(0, T^*)$. This solution satisfies an energy inequality, which implies its uniqueness in that class.

Linearised problems. We study two linearised problems, one for the velocity $\boldsymbol{v}$, and the other for $\boldsymbol{A}$. We first recall, without proof, some well known results (see, for example, [14]) for the time-dependent Stokes problem

$$\left.\begin{aligned} & Re\,\boldsymbol{v}' + \gamma\boldsymbol{\mathcal{L}}\boldsymbol{v} = \boldsymbol{F} \\ & \operatorname{div}\boldsymbol{v} = 0 \\ & \boldsymbol{v}(0) = \boldsymbol{v}_0 \end{aligned}\right\} \tag{3.7}$$

where $\boldsymbol{F}$ is a given body force.

Lemma 1 *Assume that* $\Gamma \in C^2$, $\boldsymbol{v}_0 \in \mathbb{V}$ *and* $\boldsymbol{F} \in \mathbb{L}^2(\Omega_T)$. *If* $\gamma > 0$, *then the Stokes problem (3.7) admits a unique solution* $\boldsymbol{v} \in \mathbb{L}^2(0,T;D(\boldsymbol{\mathcal{L}})) \cap C([0,T],\mathbb{V})$ *such that* $\boldsymbol{v}' \in \mathbb{L}^2(\Omega_T)$ *and* $p \in L^2(0,T;H^1)$. *Furthermore, there exists a constant* $C_1(Re,\gamma,\Omega)$ *such that*

$$\begin{aligned} & ||\boldsymbol{v}||^2_{\mathbb{L}^2(0,T;D(\boldsymbol{\mathcal{L}}))\cap\mathbb{L}^\infty(0,T;\mathbb{V})} + ||\boldsymbol{v}'||^2_{\mathbb{L}^2(\Omega_T)} + ||p||^2_{\mathbb{L}^2(0,T;H^1)} \\ & \qquad \le C_1(||\boldsymbol{v}_0||^2_{\mathbb{L}^2(\Omega_T)} + ||\boldsymbol{F}||^2_{\mathbb{L}^2(\Omega_T)}). \end{aligned} \tag{3.8}$$

Lemma 2 *Assume that* $\Gamma \in C^3$, $\boldsymbol{F}' \in \mathbb{L}^2(0,T,\mathbb{H}^{-1})$, $\boldsymbol{v}_0 \in D(\boldsymbol{\mathcal{L}})$.
If $\gamma \ge 0$, *then the unique solution of problem (3.7) satisfies*

$$\begin{aligned} & \boldsymbol{v} \in \mathbb{L}^2(0,T;\mathbb{H}^3) \cap C([0,T];D(\boldsymbol{\mathcal{L}})), \\ & \boldsymbol{v}' \in \mathbb{L}^2(0,T;\mathbb{V}) \cap C([0,T];\mathbb{H}), \\ & p \in L^2(0,T;H^2), \end{aligned}$$

and there exists a constant $C_2(Re,\gamma,\Omega)$ *such that*

$$\begin{aligned} & ||\boldsymbol{v}||^2_{\mathbb{L}^2(0,T;\mathbb{H}^3)\cap\mathbb{L}^\infty(0,T;D(\boldsymbol{\mathcal{L}}))} + ||\boldsymbol{v}'||^2_{\mathbb{L}^2(0,T;\mathbb{V})\cap\mathbb{L}^\infty(0,T;\mathbb{H})} + ||p||^2_{\mathbb{L}^2(0,T;\mathbb{H}^2)} \\ & \le C_2\left\{|\boldsymbol{\mathcal{L}}\boldsymbol{v}_0|^2 + ||\boldsymbol{F}||^2_{\mathbb{L}^1(0,T;\mathbb{H}^1)} + ||\boldsymbol{F}'||^2_{\mathbb{L}^1(0,T;\mathbb{H}^{-1})} + |\boldsymbol{F}(0)|^2\right\}. \end{aligned} \tag{3.9}$$

We now turn to the study of a linearised problem associated with the equation (3.5) for $\boldsymbol{A}$. For a given velocity field $\overline{\boldsymbol{v}}$, we first show the existence and the uniqueness of a regular solution $\boldsymbol{A}$ to the problem

$$We\{\boldsymbol{A}' + (\overline{\boldsymbol{v}} \cdot \nabla)\boldsymbol{A} + \boldsymbol{A}\overline{\boldsymbol{W}} - \overline{\boldsymbol{W}}\boldsymbol{A} - 3\bar{\lambda}(\boldsymbol{A}\overline{\boldsymbol{D}} + \overline{\boldsymbol{D}}\boldsymbol{A})\} + \boldsymbol{A}$$
$$= 2\omega\overline{\boldsymbol{D}} - 2\bar{\lambda}We(\boldsymbol{A} : \overline{\boldsymbol{D}})\boldsymbol{I}, \qquad (3.10)$$

$$\boldsymbol{A}(0) = \boldsymbol{A}_0 \qquad \text{a.e in } \Omega, \qquad (3.11)$$

where

$$\overline{\boldsymbol{D}} = \frac{1}{2}\left(\nabla\overline{\boldsymbol{v}} + (\nabla\overline{\boldsymbol{v}})^T\right) \quad \text{and} \quad \overline{\boldsymbol{W}} = \frac{1}{2}\left(\nabla\overline{\boldsymbol{v}} - (\nabla\overline{\boldsymbol{v}})^T\right).$$

Lemma 3 *Assume that* $\Gamma \in C^1$, $\overline{\boldsymbol{v}} \in \mathbb{L}^1(0,T;\mathbb{H}^3) \cap D(\mathcal{L})$, $\boldsymbol{A}_0 \in \mathbb{H}^2(\Omega)$. *Then the problem (3.10) admits a unique solution* $\boldsymbol{A} \in C([0,T],\mathbb{H}^2)$. *Futhermore, there exists a constant* $C(\Omega,\omega,We)$ *such that*

$$\|\boldsymbol{A}\|_{\mathbb{L}^\infty(0,T;\mathbb{H}^2)} \leq \left(\|\boldsymbol{A}_0\|_{\mathbb{H}^2} + \frac{2\omega}{We}\right)\exp(C\|\overline{\boldsymbol{v}}\|_{\mathbb{L}^1(0,T;\mathbb{H}^3)}). \qquad (3.12)$$

In addition, if $\overline{\boldsymbol{v}} \in C([0,T],D(\mathcal{L}))$, *then* $\boldsymbol{A}' \in C([0,T],\mathbb{H}^1)$ *and satisfies*

$$\|\boldsymbol{A}'\|_{\mathbb{L}^\infty(0,T;\mathbb{H}^1)} \leq$$
$$C\left(\|\overline{\boldsymbol{v}}_{\mathbb{L}^1(0,T;\mathbb{H}^3)} + \frac{1}{We}\right)\left(\|\boldsymbol{A}_0\|_{\mathbb{L}^1(0,T;\mathbb{H}^2)} + \frac{2\omega}{We}\right)\exp(C\|\overline{\boldsymbol{v}}\|_{\mathbb{L}^1(0,T;\mathbb{H}^3)}). \qquad (3.13)$$

Proof

The proof of existence of a unique solution follows directly by application of the method of characteristics [5].

We turn now to the derivation of the estimates (3.12) and (3.13).

Estimate (3.12) We obtain three equations by, first, taking the product of (3.10) with $\boldsymbol{A}$ in $\mathbb{L}^2(\Omega)$; next, by applying the operator $\partial/\partial x_l$ to (3.10), and taking the $\mathbb{L}^2$-inner product of this equation with $\boldsymbol{A}_{,l}$; and finally, by applying the operator $\partial^2/\partial x_l \partial x_m$ to (3.10) and taking the $\mathbb{L}^2$-inner product with $\boldsymbol{A}_{,lm}$. Here subscripts following a comma denote partial derivatives with respect to the corresponding spatial variable. We add these three equations and simplify, to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(We\|\boldsymbol{A}\|^2_{\mathbb{H}^2}\right) + \|\boldsymbol{A}\|^2_{\mathbb{H}^2} = 2\omega(\overline{\boldsymbol{D}},\boldsymbol{A})_{\mathbb{H}^2} + We(\overline{\boldsymbol{W}}\boldsymbol{A} - \boldsymbol{A}\overline{\boldsymbol{W}},\boldsymbol{A})_{\mathbb{H}^2}\\
&-We\int_{\Omega}\{\overline{v}_{k,l}\left[\boldsymbol{A}_{,k}:\boldsymbol{A}_{,l} + 2\boldsymbol{A}_{,km}:\boldsymbol{A}_{,lm}\right] + \overline{v}_{k,lm}\boldsymbol{A}_{,k}:\boldsymbol{A}_{,lm}\}\,dx\\
&-2\bar{\lambda}We\left(((\boldsymbol{A}:\overline{\boldsymbol{D}})\boldsymbol{I},\boldsymbol{A})_{\mathbb{H}^2} + 3\bar{\lambda}We(\boldsymbol{A}\overline{\boldsymbol{D}} + \overline{\boldsymbol{D}}\boldsymbol{A},\boldsymbol{A})_{\mathbb{H}^2}\right).
\end{aligned}\tag{3.14}$$

The right-hand side of (3.14) may be estimated as in [8], and we deduce that

$$\frac{1}{2}\frac{d}{dt}(We\|\boldsymbol{A}\|^2_{\mathbb{H}^2}) + \|\boldsymbol{A}\|^2_{\mathbb{H}^2} \leq 2\omega C_0\|\overline{\boldsymbol{v}}\|_{\mathbb{H}^3}\|\boldsymbol{A}\|_{\mathbb{H}^2} + 2We\,C_0\|\overline{\boldsymbol{v}}\|_{\mathbb{H}^3}\|\boldsymbol{A}\|^2_{\mathbb{H}^2}. \tag{3.15}$$

The constant C_0 entering the estimate (3.15) depends only on Ω, and on the material constants. Inequality (3.15) implies in particular that

$$\frac{d}{dt}\left(\|\boldsymbol{A}\|_{\mathbb{H}^2}\right) \leq \frac{2\omega C_0}{We}\|\overline{\boldsymbol{v}}\|_{\mathbb{H}^3} + 2C_0\|\overline{\boldsymbol{v}}\|_{\mathbb{H}^3}\|\boldsymbol{A}\|_{\mathbb{H}^2} - \frac{1}{We}\|\boldsymbol{A}\|_{\mathbb{H}^2}. \tag{3.16}$$

Now, for a given $\alpha \in \mathbb{R}_+$, we deduce from (3.16) that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left[(\|\boldsymbol{A}\|_{\mathbb{H}^2} + \alpha)^2\right] &= (\|\boldsymbol{A}\|_{\mathbb{H}^2} + \alpha)\frac{d}{dt}(\|\boldsymbol{A}\|_{\mathbb{H}^2})\\
&\leq 2C\|\overline{\boldsymbol{v}}\|_{\mathbb{H}^3}(\|\boldsymbol{A}(t)\|_{\mathbb{H}^2} + \alpha)\left(\|\boldsymbol{A}\|_{\mathbb{H}^2} + \frac{2\omega}{We}\right)
\end{aligned}\tag{3.17}$$

for some positive constant C. We set $\alpha = 2\omega/We$; it follows from inequality (3.17) that

$$\frac{1}{2}\frac{d}{dt}\left[\left(\|\boldsymbol{A}(t)\|_{\mathbb{H}^2} + \frac{2\omega}{We}\right)^2\right] \leq 2C\|\overline{\boldsymbol{v}}(t)\|_{\mathbb{H}^3}\left(\|\boldsymbol{A}(t)\|_{\mathbb{H}^2} + \frac{2\omega}{We}\right)^2. \tag{3.18}$$

Integration of (3.18) over $(0,t) \subset (0,T)$ gives (3.12). Therefore $\boldsymbol{A}$ belongs to $\mathbb{L}^\infty(0,T,\mathbb{H}^2)$.

Estimate (3.13) Let us first consider the initial-value problem

$$\begin{cases} \dfrac{d}{dt}\boldsymbol{A}(t,\boldsymbol{x}) = \boldsymbol{G}(t,\boldsymbol{A}) \quad in \quad \Omega_T,\\ \boldsymbol{A}(0,\boldsymbol{x}) = \boldsymbol{A}_0 \quad \text{in} \quad \Omega, \end{cases}\tag{3.19}$$

where

$$\begin{aligned}
&\boldsymbol{G}(t,\boldsymbol{A}) = \frac{1}{We}\left(2\omega\overline{\boldsymbol{D}} + 2\bar{\lambda}We(\boldsymbol{A}:\boldsymbol{D})\boldsymbol{I} - \boldsymbol{A}\right) - (\overline{\boldsymbol{v}}\cdot\nabla)\boldsymbol{A}\\
&-(\boldsymbol{A}\overline{\boldsymbol{W}} - \overline{\boldsymbol{W}}\boldsymbol{A}) + 3\bar{\lambda}(\boldsymbol{A}\overline{\boldsymbol{D}} + \overline{\boldsymbol{D}}\boldsymbol{A}).
\end{aligned}\tag{3.20}$$

We note that the solution of (3.19) is given by

$$\begin{aligned} \boldsymbol{A}(t) &= \boldsymbol{A}_0 + \int_0^t \Big[\frac{1}{We}\left(2\omega\overline{\boldsymbol{D}} + 2\bar{\lambda}We(\boldsymbol{A}:\boldsymbol{D})\boldsymbol{I} - \boldsymbol{A}\right) \\ &\qquad -(\overline{\boldsymbol{v}}\cdot\nabla)\boldsymbol{A} - (\boldsymbol{A}\overline{\boldsymbol{W}} - \overline{\boldsymbol{W}}\boldsymbol{A}) + 3\bar{\lambda}(\boldsymbol{A}\overline{\boldsymbol{D}} + \overline{\boldsymbol{D}}\boldsymbol{A})\Big]\, ds. \end{aligned} \tag{3.21}$$

If $\boldsymbol{G} \in C([0,T],\mathbb{H}^2)$ then the solution (3.21) belongs to $C([0,T],\mathbb{H}^2)$ (at least locally). If $\overline{\boldsymbol{v}} \in C([0,T],\mathbb{H}^2)$, we have

$$\begin{aligned} \boldsymbol{A}'(t) &= \frac{1}{We}\left(2\omega\overline{\boldsymbol{D}} - 2\bar{\lambda}We(\boldsymbol{A}:\overline{\boldsymbol{D}})\boldsymbol{I} - \boldsymbol{A}\right) - (\overline{\boldsymbol{v}}\cdot\nabla)\boldsymbol{A} \\ &\quad -(\boldsymbol{A}\overline{\boldsymbol{W}} - \overline{\boldsymbol{W}}\boldsymbol{A}) + 3\bar{\lambda}(\boldsymbol{A}\overline{\boldsymbol{D}} + \overline{\boldsymbol{D}}\boldsymbol{A}) \end{aligned} \tag{3.22}$$

and the righthand side belongs to $C([0,T],\mathbb{H}^1)$. Taking the $\mathbb{H}^1$-norm of both sides of (3.22) we obtain

$$\begin{aligned} \|\boldsymbol{A}'(t)\|_{\mathbb{H}^1} &\leq C_0\frac{2\omega}{We}\|\overline{\boldsymbol{v}}\|_{\mathbb{H}^3} + \frac{1}{We}C_0\|\boldsymbol{A}(t)\|_{\mathbb{H}^2} + C_0\|\overline{\boldsymbol{v}}\|_{\mathbb{H}^3}\|\boldsymbol{A}(t)\|_{\mathbb{H}^2} \\ &\leq C_0\left(\|\overline{\boldsymbol{v}}\|_{\mathbb{H}^3} + \frac{1}{We}\right)\left(\|\boldsymbol{A}(t)\|_{\mathbb{H}^2}) + \frac{2\omega}{We}\right). \end{aligned} \tag{3.23}$$

Equation (3.23) may be written in the form

$$\|\boldsymbol{A}(t)\|_{\mathbb{H}^2} + \frac{2\omega}{We} \geq \frac{\|\boldsymbol{A}'(t)\|_{\mathbb{H}^1}}{C_0\left(\|\overline{\boldsymbol{v}}\|_{\mathbb{H}^3} + \frac{1}{We}\right)}. \tag{3.24}$$

We integrate (3.18) over $(0,T)$, and deduce that

$$\|\boldsymbol{A}\|_{\mathbb{H}^2} + \frac{2\omega}{We} \leq \left(\|\boldsymbol{A}_0\|_{\mathbb{H}^2} + \frac{2\omega}{We}\right)\exp\{C\|\overline{\boldsymbol{v}}\|_{\mathbb{L}^1(0,T;\mathbb{H}^3)}\}. \tag{3.25}$$

Next, we use (3.25) to deduce from (3.24) that

$$\begin{aligned} &\|\boldsymbol{A}'(t)\|_{\mathbb{L}^\infty(0,T;\mathbb{H}^1)} \leq \\ &C\left(\|\overline{\boldsymbol{v}}\|_{\mathbb{L}^1(0,T;\mathbb{H}^3)} + \frac{1}{We}\right)\left(\|\boldsymbol{A}_0(t)\|_{\mathbb{L}^1(0,T;\mathbb{H}^2)} + \frac{2\omega}{We}\right)\exp(C\|\overline{\boldsymbol{v}}\|_{\mathbb{L}^1(0,T;\mathbb{H}^3)}). \end{aligned}$$

Thus (3.13) is proved.

Local existence of a regular solution. Let us now consider the problem

$$\left.\begin{aligned} Re\,\boldsymbol{v}' + \bar{\gamma}\boldsymbol{\mathcal{L}}\boldsymbol{v} = \boldsymbol{b} - Re(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} \\ + \operatorname{div}\left\{\left[N(\boldsymbol{A}\boldsymbol{D}+\boldsymbol{D}\boldsymbol{A}) + \bar{N}_p(\boldsymbol{A}:\boldsymbol{D})\boldsymbol{I}\right]\right\} \\ We\{\boldsymbol{A}' + (\boldsymbol{v}\cdot\nabla)\boldsymbol{A} + \boldsymbol{A}\boldsymbol{W} - \boldsymbol{W}\boldsymbol{A} - 3\bar{\lambda}(\boldsymbol{A}\boldsymbol{D}+\boldsymbol{D}\boldsymbol{A})\} + \boldsymbol{A} \\ = 2\omega\boldsymbol{D} - 2\bar{\lambda}We(\boldsymbol{A}:\boldsymbol{D})\boldsymbol{I} \\ \\ \boldsymbol{v}(\cdot,t)\in\mathbb{V}, \quad \boldsymbol{A}(\cdot,t)\in\mathbb{H}^2(\Omega)\cap\mathbb{X} \text{ for almost all t,} \\ \boldsymbol{v}(0)=\boldsymbol{v}_0, \quad \boldsymbol{A}(0)=\boldsymbol{A}_0 \end{aligned}\right\} \tag{3.26}$$

Theorem 1 (Local existence of solution: linear closure). *Assume that* $\Gamma\in C^3, \boldsymbol{b}\in\mathbb{L}^2_{loc}(\mathbb{R}_+;\mathbb{H}^1),\ \boldsymbol{b}'\in\mathbb{L}^2_{loc}(\mathbb{R}_+;\mathbb{H}^{-1}),\ \boldsymbol{v}_0\in D(\boldsymbol{\mathcal{L}}),\ \boldsymbol{A}_0\in\mathbb{H}^2(\Omega)$. *If* $\gamma>0$, *then there exist* $T^*>0$, $\boldsymbol{v}\in\mathbb{L}^2(0,T^*;\mathbb{H}^3)\cap C([0,T^*],D(\boldsymbol{\mathcal{L}}))$, *with* $\boldsymbol{v}'\in\mathbb{L}^2(0,T^*;\mathbb{V})\cap C([0,T^*],\mathbb{H})$*;* $p\in L^2(0,T^*;\mathbb{H}^2)$ *(p is the associated pressure), and* $\boldsymbol{A}\in C([0,T^*],\mathbb{H}^2)\cap\mathbb{X}$, *such that* $(\boldsymbol{v},\boldsymbol{A},p)$ *is a solution to the problem (3.26) in* Ω_{T^*}.

Proof. Step 1. For $T>0, B_1>0$ and $B_2>0$, we define the set

$$\begin{aligned} R_T = \{\ & (\overline{\boldsymbol{v}},\overline{\boldsymbol{A}}), \overline{\boldsymbol{v}}\in C([0,T],D(\boldsymbol{\mathcal{L}}))\cap\mathbb{L}^2(0,T;\mathbb{H}^3), \\ & \overline{\boldsymbol{v}}'\in C([0,T],\mathbb{H})\cap\mathbb{L}^2(0,T;\mathbb{V}); \overline{\boldsymbol{A}}\in\mathbb{L}^\infty(0,T;\mathbb{H}^2); \\ & \overline{\boldsymbol{A}}'\in\mathbb{L}^\infty(0,T;\mathbb{H}^1); \overline{\boldsymbol{v}}(0)=\overline{\boldsymbol{v}}_0, \overline{\boldsymbol{A}}(0)=\overline{\boldsymbol{A}}_0\ \in\ \Omega, \\ & \|\overline{\boldsymbol{v}}\|^2_{\mathbb{L}^\infty(0,T;D(\boldsymbol{\mathcal{L}}))\cap\mathbb{L}^2(0,T;\mathbb{H}^3)} + \|\overline{\boldsymbol{v}}'\|^2_{\mathbb{L}^\infty(0,T;\mathbb{H})\cap\mathbb{L}^2(0,T;\mathbb{V})} \le B_1, \\ & \|\overline{\boldsymbol{A}}\|^2_{\mathbb{L}^\infty(0,T;\mathbb{H}^2)} \le B_1, \quad \|\overline{\boldsymbol{A}}'\|_{\mathbb{L}^\infty(0,T;\mathbb{H}^1)} \le B_2\ \}\ . \end{aligned} \tag{3.27}$$

We show that, if B_1 is large enough, then $\forall T>0, R_T\neq\emptyset$. Let $\boldsymbol{v}^*$ be the solution of the problem

$$\begin{aligned} & Re\,\boldsymbol{v}^{*'} + \gamma\boldsymbol{\mathcal{L}}\boldsymbol{v}^* = \boldsymbol{0} \qquad \text{a.e in } \mathbb{R}_+, \\ & \boldsymbol{v}^*(t)\in\mathbb{V} \qquad \text{in } \mathbb{R}_+, \\ & \boldsymbol{v}^*(0)=\boldsymbol{v}_0. \end{aligned}$$

If $\gamma>0$, then from Lemma 2, there exists a constant $D_1(N_p,Re,\Omega)$ such that

$$\|\boldsymbol{v}^*\|_{\mathbb{L}^2(0,T;\mathbb{H}^3)\cap\mathbb{L}^\infty(0,T;D(\boldsymbol{\mathcal{L}}))} + \|\boldsymbol{v}^{*'}\|^2_{\mathbb{L}^\infty(0,T;\mathbb{H})\cap\mathbb{L}^2(0,T;\mathbb{V})} \le D_1|\boldsymbol{\mathcal{L}}\boldsymbol{v}_0|^2. \tag{3.28}$$

If we set

$$B_1 > D_1|\boldsymbol{\mathcal{L}v}_0|^2 + \|\boldsymbol{A}_0\|_{\mathbb{H}^2}, \tag{3.29}$$

then $(\boldsymbol{v}^*, \boldsymbol{A}_0) \in R_T$, for all $T > 0$.

Next, we consider the mapping

$$\begin{aligned} &\phi : R_T \longrightarrow X_T = C([0,T];\mathbb{V}) \times C([0,T];\mathbb{H}^1), \\ &(\overline{\boldsymbol{v}}, \overline{\boldsymbol{A}}) \longmapsto (\boldsymbol{v}, \boldsymbol{A}), \end{aligned} \tag{3.30}$$

where $\boldsymbol{A}$ and $\boldsymbol{v}$ are the unique solutions of (3.7) and (3.10) respectively, and with

$$\boldsymbol{F} = -Re(\overline{\boldsymbol{v}} \cdot \nabla)\overline{\boldsymbol{v}} + \boldsymbol{b} + \operatorname{div}\left\{\left[N(\overline{\boldsymbol{A}}\ \overline{\boldsymbol{D}} + \overline{\boldsymbol{D}}\ \overline{\boldsymbol{A}}) + \bar{N}_p(\overline{\boldsymbol{A}} : \overline{\boldsymbol{D}})\boldsymbol{I}\right]\right\}. \tag{3.31}$$

Clearly a fixed point of ϕ is a solution to the problem (3.26).

Step 2. We show that ϕ satisfies the conditions of Schauder's Fixed Point Theorem [13]. First we show that there exists T^* such that $\phi(R_{T^*}) \subset R_{T^*}$. If $(\overline{\boldsymbol{v}}, \overline{\boldsymbol{A}}) \in R_T$, then from (3.31) and the Poincaré-Friedrichs inequality, we obtain

$$\|\boldsymbol{F}\|^2_{\mathbb{L}^2(0,T;\mathbb{H}^1)} \leq \int_0^T \left[D_2(\|\overline{\boldsymbol{v}}\|^4_{\mathbb{H}^3} + \|\overline{\boldsymbol{A}}\|^2_{\mathbb{H}^2}\|\overline{\boldsymbol{v}}\|^2_{\mathbb{H}^3}) + \|\boldsymbol{b}\|^2_{\mathbb{H}^1}\right] dt,$$

where D_2 depends on Ω, N_p, and N_s. Thus

$$\|\boldsymbol{F}\|^2_{\mathbb{L}^2(0,T;\mathbb{H}^1)} \leq D_2 B_1^2 T + \|\boldsymbol{b}\|^2_{\mathbb{L}^2(0,T;\mathbb{H}^1)}. \tag{3.32}$$

In the same way we find that

$$|\boldsymbol{F}(0)|^2 \leq D_2\left(|\boldsymbol{\mathcal{L}v}_0|^2 + |\boldsymbol{A}_0|^2 + 1\right)|\boldsymbol{\mathcal{L}v}_0|^2 + |\boldsymbol{b}(0)|^2 \tag{3.33}$$

and

$$\begin{aligned} \|\boldsymbol{F}'\|^2_{\mathbb{L}^2(0,T;\mathbb{H}^{-1})} \leq& \int_0^T D_2\left[\|\overline{\boldsymbol{v}}\|^2_{\mathbb{H}^3}\|\overline{\boldsymbol{v}}'\|^2_{\mathbb{H}^2} + \|\overline{\boldsymbol{A}}'\|^2_{\mathbb{H}^1}\|\overline{\boldsymbol{v}}\|^2_{\mathbb{H}^3}\right] dt \\ &+ \int_0^T \left[D_2(\|\overline{\boldsymbol{A}}\|^2_{\mathbb{H}^2}\|\overline{\boldsymbol{v}}'\|^2_{\mathbb{H}^2}) + \|\boldsymbol{b}\|^2_{\mathbb{H}^{-1}}\right] dt, \end{aligned}$$

so that

$$\|\boldsymbol{F}'\|^2_{\mathbb{L}^2(0,T;\mathbb{H}^{-1})} \leq D_2(B_1^2 + B_1 B_2^2)T + \|\boldsymbol{b}'\|^2_{\mathbb{L}^2(0,T;\mathbb{H}^{-1})}. \tag{3.34}$$

From Lemmas 2 and 3, we use (3.32)–(3.34) to get

$$\begin{aligned} & \|\boldsymbol{v}\|^2_{\mathbb{L}^2(0,T;\mathbb{H}^3)\cap\mathbb{L}^2(0,T;D(\mathcal{L}))} + \|\boldsymbol{v}'\|^2_{\mathbb{L}^2(0,T;\mathbb{H}^{-1})\cap\mathbb{L}^\infty(0,T;\mathbb{H})} \\ \leq\ & C_2\left[|\mathcal{L}\boldsymbol{v}_0|^2 + \|\boldsymbol{F}\|^2_{\mathbb{L}(0,T;\mathbb{H}^1)} + \|\boldsymbol{F}'\|^2_{\mathbb{L}(0,T;\mathbb{H}^{-1})} + \|\boldsymbol{F}(0)\|^2\right] \\ \leq\ & C_2\Big[B_1D_2(2B_1+B_2^2)T + \|\boldsymbol{b}'\|^2_{\mathbb{L}(0,T;\mathbb{H}^{-1})} + |\boldsymbol{b}(0)|^2 \\ & \quad + D_2(|\mathcal{L}\boldsymbol{v}_0|^2 + |\boldsymbol{A}_0| + 1)|\mathcal{L}\boldsymbol{v}_0|^2\Big], \end{aligned} \tag{3.35}$$

$$\|\boldsymbol{A}\|_{\mathbb{L}^\infty(0,T;\mathbb{H}^2)} \ \leq\ \left(\|\boldsymbol{A}_0\|_{\mathbb{L}^2(0,T;\mathbb{H}^2)} + \frac{2\omega}{We}\right)\exp(C_1TB_1^{1/2}), \tag{3.36}$$

and

$$\|\boldsymbol{A}'\|_{\mathbb{L}^\infty(0,T;\mathbb{H}^1)} \ \leq\ C_0\left(B_1^{1/2} + \frac{1}{We}\right)\left(\|\boldsymbol{A}_0\|_{\mathbb{L}^2(0,T;\mathbb{H}^2)} + \frac{2\omega}{We}\right)\exp(C_1TB_1^{1/2}). \tag{3.37}$$

Therefore $(\boldsymbol{v},\boldsymbol{A})\in R_{T^*}$, if we choose T^* such that the righthand sides of (3.35)–(3.37) are bounded respectively by B_1, B_1 and B_2. We make use of (3.29), and then choose B_1, B_2 and T^* such that

$$\begin{aligned} B_1 \ \geq\ & \max\Big\{D_1|\mathcal{L}\boldsymbol{v}_0|^2, \left(\|\boldsymbol{A}_0\| + \frac{2\omega}{We}\right)e, 2C_2\Big[\|\boldsymbol{b}'\|^2_{\mathbb{L}^1(0,T;\mathbb{H}^1)} + \\ & |\boldsymbol{b}(0)|^2 \ + D_2(|\mathcal{L}\boldsymbol{v}_0|^2 + |\boldsymbol{A}_0|^2 + 1)|\mathcal{L}\boldsymbol{v}_0|^2\Big]\Big\}, \end{aligned} \tag{3.38}$$

$$B_2 \geq eC_0\left(B_1^{1/2} + \frac{1}{We}\right)\left(\|\boldsymbol{A}_0\|_{\mathbb{L}^1(0,T;\mathbb{H}^2)} + \frac{2\omega}{We}\right), \tag{3.39}$$

and

$$T^* \leq T \leq \min\left\{\frac{1}{4C_2D_2(2B_1+B_2^2)}\ ,\ \frac{1}{C_1B_1^{1/2}}\right\}. \tag{3.40}$$

Therefore, we have defined constants B_1 and B_2, depending on We, Ω, ω and on the data, and we have defined a time, say T^*, satisfying (3.40), depending on B_1, B_2 and on the data, such that $\phi(R_{T^*}) \subset R_{T^*}$.

Obviously ϕ is continuous, and by the Arzela-Ascoli Theorem, we deduce that R_{T^*} is compact in $X_T = C([0,T];\mathbb{V}) \times C([0,T];\mathbb{H}^1)$. Therefore, since R_{T^*} is a non-empty convex subset of X_T, we deduce from the Schauder Fixed Point Theorem that ϕ has a fixed point, $(\boldsymbol{v},\boldsymbol{A})$ say, which is the solution to the problem(3.26).

Step 3. It remains to show that $\boldsymbol{A} \in \mathbb{X}$; that is that $\boldsymbol{A}^T = \boldsymbol{A}$ and tr$\boldsymbol{A}$=0.

$\boldsymbol{A}^T = \boldsymbol{A}$ We take the transpose of (3.10), to obtain

$$\begin{aligned} &We\left\{\boldsymbol{A}^{T'} + (\overline{\boldsymbol{v}}\cdot\nabla)\boldsymbol{A}^T + \boldsymbol{A}^T\overline{\boldsymbol{W}} - \overline{\boldsymbol{W}}\boldsymbol{A}^T - 3\bar{\lambda}(\boldsymbol{A}^T\overline{\boldsymbol{D}} + \overline{\boldsymbol{D}}\boldsymbol{A}^T)\right\} \\ &+\boldsymbol{A}^T = 2\omega\overline{\boldsymbol{D}} - 2\bar{\lambda}We(\boldsymbol{A}:\overline{\boldsymbol{D}})\boldsymbol{I}. \end{aligned} \tag{3.41}$$

Next, we set $\boldsymbol{Q} = \boldsymbol{A}^T - \boldsymbol{A}$, and substract (3.10) from (3.41) to get

$$We\{\boldsymbol{Q}' + (\overline{\boldsymbol{v}}\cdot\nabla)\boldsymbol{Q} + \boldsymbol{Q}\overline{\boldsymbol{W}} - \overline{\boldsymbol{W}}\boldsymbol{Q} - 3\bar{\lambda}(\boldsymbol{Q}\overline{\boldsymbol{D}} + \overline{\boldsymbol{D}}\boldsymbol{Q})\} + \boldsymbol{Q} = \boldsymbol{0}. \tag{3.42}$$

We now take the $\mathbb{L}^2$-inner product of (3.42) with $\boldsymbol{Q}$, to obtain

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{Q}\|^2 + (\boldsymbol{Q}\overline{\boldsymbol{W}} - \overline{\boldsymbol{W}}\boldsymbol{Q}, \boldsymbol{Q}) - 3\bar{\lambda}(\boldsymbol{Q}\overline{\boldsymbol{D}} + \overline{\boldsymbol{D}}\boldsymbol{Q}, \boldsymbol{Q}) + \frac{1}{We}\|\boldsymbol{Q}\|^2 = 0. \tag{3.43}$$

Some terms in (3.43) are now simplified. Using the identity $\boldsymbol{AB}:\boldsymbol{C}=\boldsymbol{B}:\boldsymbol{A}^T\boldsymbol{C}$ and the skew-symmetry of $\overline{\boldsymbol{W}}$, we have

$$(\boldsymbol{Q}\overline{\boldsymbol{W}} - \overline{\boldsymbol{W}}\boldsymbol{Q}, \boldsymbol{Q}) = \int_\Omega (\boldsymbol{Q}^T\boldsymbol{Q} : \overline{\boldsymbol{W}} - \boldsymbol{Q}\boldsymbol{Q}^T : \overline{\boldsymbol{W}})\, dx = 0$$

and

$$(\boldsymbol{Q}\overline{\boldsymbol{D}} + \overline{\boldsymbol{D}}\boldsymbol{Q}, \boldsymbol{Q}) \leq 2\|\boldsymbol{Q}\|^2_{\mathbb{L}^2}\|\overline{\boldsymbol{v}}\|_{\mathbb{H}^3}.$$

Using the above results, we deduce from (3.43) the inequality

$$\frac{d}{dt}\|\boldsymbol{Q}\|^2_{\mathbb{L}^2} \leq \left(C\|\overline{\boldsymbol{v}}\|_{\mathbb{H}^3} + \frac{1}{We}\right)\|\boldsymbol{Q}\|^2_{\mathbb{L}^2}.$$

Since $\overline{\boldsymbol{v}} \in \mathbb{L}^2(0,T^*;\mathbb{H}^3)$ we may apply Gronwall's Lemma to conclude that $\boldsymbol{Q}=\boldsymbol{0}$, or $\boldsymbol{A}^T=\boldsymbol{A}$.

tr$\boldsymbol{A}$=0 First we recall that tr$\overline{\boldsymbol{A}}$=tr$\boldsymbol{A}_0$=0. We set Z=tr$\boldsymbol{A}$, and take the trace of both sides of (3.10), to obtain

$$We\{Z' + (\overline{\boldsymbol{v}}\cdot\nabla)Z - 2\bar{\lambda}(\boldsymbol{A}:\overline{\boldsymbol{D}})\} + Z = -2\bar{\lambda}We(\boldsymbol{A}:\overline{\boldsymbol{D}}).$$

This implies that

$$Z' + (\overline{\boldsymbol{v}}\cdot\nabla)Z + KZ = 0. \tag{3.44}$$

Clearly, $Z_1(t) = 0$ solves (3.44). Now assume that Z_2 is also a solution of (3.44); this is a linear differential equation, so that if we set $Z = Z_1 - Z_2$, then it follows, after taking the L^2-inner product of the resulting equation with Z, that

$$\frac{d}{dt}|Z|^2 + K|Z|^2 = 0.$$

We now apply Gronwall's Lemma and conclude that $Z = 0$; that is $Z_1 = Z_2$. This completes the proof of the theorem. □

3.1.1 Uniqueness of the solution

We now show that the solution obtained in Theorem 1 is the only one in the class of regular solutions.

Theorem 2 *Let $T^* > 0$. The problem (3.26) admits at most one solution $(\boldsymbol{v},\boldsymbol{A})$ in $\mathbb{L}^2(0,T;\mathbb{H}^3) \cap C([0,T];D(\boldsymbol{\mathcal{L}})) \times C([0,T];\mathbb{H}^2)$. The pressure p is unique up to an additive constant in $L^2(0,T;\mathbb{H}^2)$*

Proof. As usual, we take the difference of two solutions $(\boldsymbol{v}_1, \boldsymbol{A}_1, p_1)$ and $(\boldsymbol{v}_2, \boldsymbol{A}_2, p_2)$, corresponding to the same data. Set $\boldsymbol{v}{=}\boldsymbol{v}_1 - \boldsymbol{v}_2$ and $\boldsymbol{A}{=}\boldsymbol{A}_1 - \boldsymbol{A}_2$. The functions $\boldsymbol{v}$ and $\boldsymbol{A}$ satisfy the equations

$$\begin{aligned}
&Re\{\boldsymbol{v}' + (\boldsymbol{v}_1\cdot\nabla)\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}_2\} + \bar{\gamma}\boldsymbol{\mathcal{L}}\boldsymbol{v} = \mathrm{div}(\boldsymbol{S}_1 - \boldsymbol{S}_2),\\
&We\{\boldsymbol{A}' + (\boldsymbol{v}\cdot\nabla)\boldsymbol{A}_1 + (\boldsymbol{v}_2\cdot\nabla)\boldsymbol{A}\} + \boldsymbol{A} = 2\omega\boldsymbol{D} \qquad (3.45)\\
&-2\bar{\lambda}We(\boldsymbol{A}_1 : \boldsymbol{D}_1 - \boldsymbol{A}_2 : \boldsymbol{D}_2)\boldsymbol{I} - We\left(R(\boldsymbol{A}_1, \boldsymbol{v}) + R(\boldsymbol{A}, \boldsymbol{v}_2)\right),
\end{aligned}$$

where

$$\begin{aligned}
\boldsymbol{S}_i &= N(\boldsymbol{A}_i\boldsymbol{D}_i + \boldsymbol{D}_i\boldsymbol{A}_i) + \bar{N}_p(\boldsymbol{A}_i : \boldsymbol{D}_i)\boldsymbol{I},\\
R(\boldsymbol{A}, \boldsymbol{v}) &= \boldsymbol{A}\boldsymbol{W} - \boldsymbol{W}\boldsymbol{A} - 3\bar{\lambda}(\boldsymbol{A}\boldsymbol{D} + \boldsymbol{D}\boldsymbol{A}). \qquad (3.46)
\end{aligned}$$

Now, we take the $\mathbb{L}^2$-inner product of (3.45) with $\boldsymbol{v}$ and $\boldsymbol{A}$ respectively, and integrate over Ω, to obtain

$$Re\left\{\frac{1}{2}\frac{d}{dt}\|\boldsymbol{v}(t)\|^2_{\mathbb{L}^2} + ((\boldsymbol{v}\cdot\nabla)\boldsymbol{v}_2, \boldsymbol{v})\right\} + \bar{\gamma}\|\boldsymbol{v}(t)\|^2_{\mathbb{H}^1} = -\left(\boldsymbol{S}_1 - \boldsymbol{S}_2, \boldsymbol{D}\right),$$

$$
\begin{aligned}
&We\left\{\frac{1}{2}\frac{d}{dt}\|\boldsymbol{A}(t)\|^2_{\mathbb{L}^2}+((\boldsymbol{v}\cdot\nabla)\boldsymbol{A}_1,\boldsymbol{A})\right\}+|\boldsymbol{A}(t)|^2_{\mathbb{L}^2}=2\omega(\boldsymbol{D},\boldsymbol{A}(t)) \\
&-We\left(R(\boldsymbol{A}_1,\boldsymbol{v})+R(\boldsymbol{A},\boldsymbol{v}_2),\boldsymbol{A}(t)\right).
\end{aligned}
\tag{3.47}
$$

Using (3.46) and the Sobolev inequality we obtain the estimates

$$
\begin{aligned}
|(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}_2| &\leq C_0\|\boldsymbol{v}_2\|_{\mathbb{H}^3}|\boldsymbol{v}|, \\
|R(\boldsymbol{A},\boldsymbol{v}_2)| &\leq C_0\|\boldsymbol{v}_2\|_{\mathbb{H}^3}|\boldsymbol{A}|, \\
|(\boldsymbol{v}\cdot\nabla)\boldsymbol{A}_1+R(\boldsymbol{A}_1,\boldsymbol{v})| &\leq C_0\|\boldsymbol{A}_1\|_{\mathbb{H}^2}\|\boldsymbol{v}\|_{\mathbb{H}^2}.
\end{aligned}
\tag{3.48}
$$

Also,

$$
(\boldsymbol{S}_1-\boldsymbol{S}_2,\boldsymbol{D})=\frac{2\mu_I}{7\mu}N\left(\boldsymbol{A}_1\boldsymbol{D}+\boldsymbol{D}\boldsymbol{A}_1+\boldsymbol{A}\boldsymbol{D}^2+\boldsymbol{D}^2\boldsymbol{A},\boldsymbol{D}\right). \tag{3.49}
$$

We make use of (3.48) and (3.49), we add $(3.47)_1$ and $(3.47)_2$, and use Young's inequality to deduce the energy inequality

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(Re\|\boldsymbol{v}\|^2_{\mathbb{L}^2}+We\|\boldsymbol{A}\|^2_{\mathbb{L}^2}\right)+\left[\bar{\gamma}-C_0\frac{\epsilon}{2}\|\boldsymbol{A}_1\|_{\mathbb{H}^2}\right]\|\boldsymbol{v}\|^2_{\mathbb{H}^2} \\
&-C_0\frac{\epsilon}{2}\left[\|\boldsymbol{v}\|^2_{\mathbb{H}^3}+2\omega+We\|\boldsymbol{A}_1\|_{\mathbb{H}^2}\right]\|\boldsymbol{v}\|^2_{\mathbb{H}^2} \\
&\leq\frac{1}{2\epsilon}\left(C_0\|\boldsymbol{v}_2\|_{\mathbb{H}^3}+2\epsilon+2\omega C_0+C_0We\|\boldsymbol{A}_1\|_{\mathbb{H}^2}+2\epsilon We\|\boldsymbol{v}\|^2_{\mathbb{H}^3}\right)\|\boldsymbol{A}\|^2_{\mathbb{L}^2} \\
&+\frac{1}{2\epsilon}\left(C_0\|\boldsymbol{A}_1\|_{\mathbb{H}^2}+2\epsilon Re\|\boldsymbol{v}_2\|_{\mathbb{H}^3}\right)\|\boldsymbol{v}\|^2_{\mathbb{L}^2} \\
&\leq\left(Re\|\boldsymbol{v}\|^2_{\mathbb{L}^2}+We|\boldsymbol{A}|^2\right)\left(C_0\|\boldsymbol{v}_2\|_{\mathbb{H}^3}+k\right),
\end{aligned}
\tag{3.50}
$$

where

$$
k=\frac{1}{2\epsilon}\left(2\epsilon Re\|\boldsymbol{v}_2\|_{\mathbb{H}^3}+C_0\|\boldsymbol{A}_1\|_{\mathbb{H}^2}+2\epsilon+2\omega C_0+C_0We+2\epsilon We\|\boldsymbol{v}_2\|^2_{\mathbb{H}^3}\right). \tag{3.51}
$$

Equation (3.51) is true for any $\epsilon>0$; we choose ϵ small enough, for example,

$$
\epsilon=\frac{\bar{\gamma}}{C_0(\|\boldsymbol{A}_1\|_{\mathbb{H}^2}+\|\boldsymbol{v}_2\|_{\mathbb{H}^3}+2\omega+We\|\boldsymbol{A}_1\|_{\mathbb{H}^2})},
$$

such that the coefficient of $\|\boldsymbol{v}\|^2_{\mathbb{H}^2}$ on the left hand side of (3.50) is positive (this requires also that $N_p<35/2$). Then (3.50) reads

$$
\frac{1}{2}\frac{d}{dt}\left(Re\|\boldsymbol{v}\|^2_{\mathbb{L}^2}+We\|\boldsymbol{A}\|^2_{\mathbb{L}^2}\right)\leq K\left(Re\|\boldsymbol{v}\|^2_{\mathbb{L}^2}+We\|\boldsymbol{A}\|^2_{\mathbb{L}^2}\right) \tag{3.52}
$$

with

$$
K=C_0\|\boldsymbol{v}_2\|_{\mathbb{H}^3}+k.
$$

We deduce from (3.52) and Gronwall's Lemma that $Re\|\boldsymbol{v}\|^2_{\mathbb{L}^2}+We\|\boldsymbol{A}\|^2_{\mathbb{L}^2}=0$, and hence also that the pressure p is constant. □

3.2 The quadratic closure approximation

Now we turn to the the quadratic closure approximation for the problem studied in Section 3.1. We prove the local existence of the solution, as well its uniqueness.

Dimensionless and traceless problem. We use the decomposition (3.2) and (3.3), and again write $\boldsymbol{A}$ for $\hat{\boldsymbol{A}}$ and use $\mathrm{tr}\boldsymbol{D} = 0$ to obtain

$$\begin{aligned} \mathcal{A}\boldsymbol{D} &= (\boldsymbol{A}+\boldsymbol{A}^*)\otimes(\boldsymbol{A}+\boldsymbol{A}^*)\boldsymbol{D} \\ &= (\boldsymbol{A}:\boldsymbol{D})(\boldsymbol{A}+\frac{1}{d}\boldsymbol{I}). \end{aligned} \tag{3.53}$$

We also set

$$\eta = \frac{1}{dD_r}\frac{V}{L}. \tag{3.54}$$

The system of dimensionless equations corresponding to the quadratic closure is therefore

$$\begin{aligned} &Re(\boldsymbol{v}' + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}) + \nabla p - \frac{2}{7}\left(1 + 2\frac{N_s}{d}\right)\Delta\boldsymbol{v} = \boldsymbol{b} + \mathrm{div}\,\boldsymbol{S}_Q, \\ &\eta\,[\boldsymbol{A}' + (\boldsymbol{v}\cdot\nabla)\boldsymbol{a} + (\boldsymbol{A}\boldsymbol{W} - \boldsymbol{W}\boldsymbol{A}) - \lambda(\boldsymbol{A}\boldsymbol{D} + \boldsymbol{D}\boldsymbol{A})] + \boldsymbol{A} \\ &= 2\lambda\eta\boldsymbol{D} - 2\lambda\eta(\boldsymbol{A}:\boldsymbol{D})(\boldsymbol{A} + \frac{1}{d}\boldsymbol{I}), \end{aligned} \tag{3.55}$$

where

$$\boldsymbol{S}_Q(\boldsymbol{A},\boldsymbol{v}) = \frac{2}{7}\left[N_p(\boldsymbol{A}:\boldsymbol{D})(\boldsymbol{A}+\frac{1}{d}\boldsymbol{I}) + N_s(\boldsymbol{A}\boldsymbol{D}+\boldsymbol{D}\boldsymbol{A})\right]. \tag{3.56}$$

Remark. Lemmas 1 and 2 are still valid with γ replaced by $\bar{\gamma} := \frac{2}{7}(1+\frac{2N_s}{d})$, but Lemma 3 will be modified appropriately below.

The linearised problem. For a given admissible velocity field $\overline{\boldsymbol{v}}$ and orientation tensor field $\overline{\boldsymbol{A}}$, let us consider the problem

$$\begin{aligned} &\eta\left[\boldsymbol{A}' + (\overline{\boldsymbol{v}}\cdot\nabla)\boldsymbol{A} + (\boldsymbol{A}\overline{\boldsymbol{W}} - \overline{\boldsymbol{W}}\boldsymbol{A}) - \lambda(\boldsymbol{A}\overline{\boldsymbol{D}} + \overline{\boldsymbol{D}}\boldsymbol{A})\right] + \boldsymbol{A} \\ &= 2\lambda\eta\overline{\boldsymbol{D}} - 2\lambda\eta(\boldsymbol{A}:\overline{\boldsymbol{D}})(\overline{\boldsymbol{A}} + \frac{1}{d}\boldsymbol{I}), \\ &\boldsymbol{A}(0) = \boldsymbol{A}_0 \quad \text{a.e in } \Omega. \end{aligned} \tag{3.57}$$

Lemma 4 *Assume that* $\Gamma \in C^1$, $\overline{\boldsymbol{v}} \in \mathbb{L}^1(0,T;\mathbb{H}^3)\cap D(\mathcal{L})$, $\overline{\boldsymbol{A}} \in \mathbb{X}\cap C([0,T],\mathbb{H}^1)$, *and* $\boldsymbol{A}_0 \in \mathbb{H}^2(\Omega)$. *Then the problem (3.57) exists a constant C(Ω,ω,We) such that*

$$||\boldsymbol{A}||_{\mathbb{L}^\infty(0,T;\mathbb{H}^2)}$$

$$\leq \left(\|\boldsymbol{A}_0\|_{\mathbb{H}^2}+1\right)\exp\left[C\left(\|\overline{\boldsymbol{v}}\|^2_{\mathbb{L}^2(0,T;\mathbb{H}^3)}\| + \|\overline{\boldsymbol{A}}\|^2_{\mathbb{L}^2(0,T;\mathbb{H}^2)} + \|\overline{\boldsymbol{v}}\|_{\mathbb{L}^1(0,T;\mathbb{H}^3)}\right)\right]. \tag{3.58}$$

In addition, if $\overline{\boldsymbol{v}} \in C([0,T], D(\mathcal{L}))$*, then* $\boldsymbol{A}' \in C([0,T],\mathbb{H}^1)$ *and satisfies*

$$\begin{aligned}\|\boldsymbol{A}'\|_{\mathbb{L}^\infty(0,T;\mathbb{H}^1)} \;\leq\;& C\left[\|\overline{\boldsymbol{v}}\|_{\mathbb{H}^3}(2\eta\lambda\|\overline{\boldsymbol{A}}\|_{\mathbb{H}^2}+\eta(2\lambda+1))+1\right]\left(\|\boldsymbol{A}_0\|_{\mathbb{H}^2}+1\right)\times \\ & \exp\left\{C\left(\|\overline{\boldsymbol{v}}\|^2_{\mathbb{L}^2(0,T;\mathbb{H}^3)}\| + \|\overline{\boldsymbol{A}}\|^2_{\mathbb{L}^2(0,T;\mathbb{H}^2)} + \|\overline{\boldsymbol{v}}\|_{\mathbb{L}^1(0,T;\mathbb{H}^3)}\right)\right\}.\end{aligned} \tag{3.59}$$

Proof. See the proof of Lemma 3.

Local existence of a regular solution. Let us consider the problem

$$\left.\begin{array}{l} Re\,\boldsymbol{v}' - \bar{\gamma}\mathcal{L}\boldsymbol{v} = \frac{2}{7}\mathrm{div}\left[N_p(\boldsymbol{A}:\boldsymbol{D})(\boldsymbol{A}+\frac{1}{d}\boldsymbol{I}) + N_s(\boldsymbol{A}\boldsymbol{D}+\boldsymbol{D}\boldsymbol{A})\right] \\ \qquad\qquad +\boldsymbol{b} - Re(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}, \\ \eta\left\{\boldsymbol{A}' + (\boldsymbol{v}\cdot\nabla)\boldsymbol{A} + (\boldsymbol{A}\boldsymbol{W}-\boldsymbol{W}\boldsymbol{A}) - \lambda(\boldsymbol{A}\boldsymbol{D}+\boldsymbol{D}\boldsymbol{A})\right\} + \boldsymbol{A} \\ = 2\lambda\eta\boldsymbol{D} - 2\lambda\eta(\boldsymbol{A}:\boldsymbol{D})(\boldsymbol{A}+\frac{1}{d}\boldsymbol{I}), \\ \\ \boldsymbol{v}(\cdot,t)\in\mathbb{V}, \quad \boldsymbol{A}(\cdot,t)\in\mathbb{H}^2(\Omega) \quad \textit{and} \quad \boldsymbol{A}(\cdot,t)\in\mathbb{X} \quad \text{for almost all } t \\ \boldsymbol{v}(0)=\boldsymbol{v}_0, \quad \boldsymbol{A}(0)=\boldsymbol{A}_0. \end{array}\right\} \tag{3.60}$$

Theorem 3 (LOCAL EXISTENCE OF SOLUTION: QUADRATIC CLOSURE). *Assume that* $\Gamma \in C^3$, $\boldsymbol{b} \in \mathbb{L}^2_{loc}(\mathbb{R}_+,\mathbb{H}^1)$, $\boldsymbol{b}' \in \mathbb{L}^2_{loc}(\mathbb{R}_+,\mathbb{H}^{-1})$, $\boldsymbol{v}_0 \in D(\mathcal{L})$, $\boldsymbol{A}_0 \in \mathbb{H}^2(\Omega)\cap\mathbb{X}$.
Then there exists $T^* > 0$*,* $\boldsymbol{v} \in \mathbb{L}^2(0,T^*;\mathbb{H}^3)\cap C([0,T^*];D(\mathcal{L}))$*, with* $\boldsymbol{v}' \in \mathbb{L}^2(0,T^*;\mathbb{V})\cap C([0,T^*;\mathbb{H})$*,* $p \in L^2(0,T^*;\mathbb{H}^2)$ *(p is the associated pressure), and* $\boldsymbol{A} \in C([0,T^*],\mathbb{H}^2)\cap\mathbb{X}$ *such that* $(\boldsymbol{v},\boldsymbol{A},p)$ *is a solution to the problem (3.60) in* Ω_{T^*}*.*

Proof. The proof of this theorem is similar to that of Theorem 1, since the difference between problems (3.26) and (3.60) lies in an extra term of the form $M(\boldsymbol{A}:\boldsymbol{D})\boldsymbol{A}$ in equations $(3.60)_1$ and $(3.60)_2$, where M is a constant. One can follow the same steps as in the proof of Theorem 1 to obtain the necessary estimates.

Uniqueness of the solution. We can show, as in Theorem 2, that the solution obtained in Theorem 3 is the only one in the class of regular solutions. The proof follows that of Theorem 2 very closely, so we merely state the result.

Theorem 4 *Let $T>0$. The problem (3.26) admits at most one solution* $(\boldsymbol{v},\boldsymbol{A})$ *in* $\mathbb{L}^2(0,T;\mathbb{H}^3)\cap C([0,T];D(\mathcal{L}))\times C([0,T];\mathbb{H}^2)$. *Furthermore, the pressure p is unique up to an additive constant in* $\mathbb{L}^2(0,T;\mathbb{H}^2)$.

4 Global existence of solutions

In this section, we show that the unique solutions obtained in Theorems 1 and 2 for the linear closure and in Theorem 3 and 4 for the quadratic closure, are defined for all $t>0$, for small enough data. These global solutions are proved to be stable in the absence of body forces.

We reiterate here the observation that it is possible, as will be shown in this Section, to obtain results on global existence for the model (2.12)-(2.13) in which the rotary diffusivity is assumed constant. Galdi and Reddy [8] have studied local existence for the case in which D_r is non-constant, and proportional to $|\boldsymbol{D}|$, and have indicated that it is not clear how their analysis may be extended to include global existence.

The approach used in this section draws substantially on the analysis by Guillopé and Saut [9].

4.1 Linear closure

We start by deriving some a priori bounds uniform in time, satisfied by the solution in Theorem 1.

Some a priori estimates. We recall first that $(\boldsymbol{v},\boldsymbol{A})$ is the unique solution to problem (3.26). We rewrite (3.15), with $\overline{\boldsymbol{v}}=\boldsymbol{v}$, and using Young's Inequality, to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(We\|\boldsymbol{A}\|^2_{\mathbb{H}^2})+\|\boldsymbol{A}\|^2_{\mathbb{H}^2}&\le 2\omega C_0\|\boldsymbol{v}\|_{\mathbb{H}^3}\|\boldsymbol{A}\|_{\mathbb{H}^2}+2We\,C_0\|\boldsymbol{v}\|_{\mathbb{H}^3}\|\boldsymbol{A}\|^2_{\mathbb{H}^2}\\ &\le 4\omega^2C_0^2\frac{\epsilon}{2}\|\boldsymbol{v}\|^2_{\mathbb{H}^3}+\frac{1}{2\epsilon}\|\boldsymbol{A}\|^2_{\mathbb{H}^2}+4We^2C_0^2\frac{\alpha}{2}\|\boldsymbol{A}\|^4_{\mathbb{H}^2}+\frac{1}{2\alpha}\|\boldsymbol{v}\|^2_{\mathbb{H}^3}.\end{aligned}$$

We choose $\epsilon = \frac{4}{3}$ and $\alpha = 3/(2\omega^2 C_0^2)$; then the previous inequality reads

$$\frac{d}{dt}(We\|\boldsymbol{A}\|^2_{\mathbb{H}^2}) + \frac{1}{2}\|\boldsymbol{A}\|^2_{\mathbb{H}^2} \leq 6\omega^2 C_0 \|\boldsymbol{v}\|^2_{\mathbb{H}^3} + \frac{6We^2}{\omega^2}\|\boldsymbol{A}\|^4_{\mathbb{H}^2}. \tag{4.1}$$

Next, we write $(3.26)_1$ in the form

$$\bar{\gamma}\boldsymbol{\mathcal{L}v} = -Re\,\boldsymbol{v}' + \boldsymbol{b} - Re(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \operatorname{div}\left\{\frac{2}{7}\left[N(\boldsymbol{AD}+\boldsymbol{DA}) + N_p(\boldsymbol{A}:\boldsymbol{D})\boldsymbol{I}\right)\right\}.$$

We take the $\mathbb{H}^1$-norm of both sides by $\boldsymbol{\mathcal{L}v}$, and use the Poincaré-Friedrichs and Young's inequalities to obtain

$$\|\boldsymbol{\mathcal{L}v}\|^2_{\mathbb{H}^1} \quad\leq\quad \frac{c_0}{\gamma^2}\left(Re^2\|\boldsymbol{v}'\|^2_{\mathbb{H}^2} + \|\boldsymbol{A}\|^2_{\mathbb{H}^2} + \|\boldsymbol{b}\|^2_{\mathbb{H}^1} + C_2 Re^2 |\boldsymbol{\mathcal{L}v}|^4\right).$$

Using Korn's Inequality, we deduce from the previous inequality an estimate for $\|\boldsymbol{v}\|_{\mathbb{H}^3}$, in the form

$$\|\boldsymbol{v}\|^2_{\mathbb{H}^3} \leq C_0\left[Re^2\|\boldsymbol{v}'\|^2_{\mathbb{H}^2} + \|\boldsymbol{A}\|^2_{\mathbb{H}^2} + \|\boldsymbol{b}\|^2_{\mathbb{H}^1} + C_2 Re^2|\boldsymbol{\mathcal{L}v}|^4\right]. \tag{4.2}$$

From (4.1) and (4.2), therefore,

$$\begin{aligned}
&\frac{d}{dt}(We\|\boldsymbol{A}\|^2_{\mathbb{H}^2}) + \left(\frac{1}{2} - 6\omega^2 C_0\right)\|\boldsymbol{A}\|^2_{\mathbb{H}^2} \\
&\leq 6\omega^2 C_0\left(Re^2\|\boldsymbol{v}'\|^2_{\mathbb{H}^2} + \|\boldsymbol{b}\|^2_{\mathbb{H}^1} + C_2 Re^2|\boldsymbol{\mathcal{L}v}|^4\right) + \frac{3We^2}{\omega^2}\|\boldsymbol{A}\|^4_{\mathbb{H}^2}.
\end{aligned} \tag{4.3}$$

Remark. Inequality (4.3) will be used only for values of D_r such that the coefficient of $\|\boldsymbol{A}\|^2_{\mathbb{H}^2}$ in the left-hand side is positive. In particular, we will choose D_r such that the coefficient of $\|\boldsymbol{A}\|^2_{\mathbb{H}^2}$ is larger than $C_0\omega^2$. That is,

$$\frac{1}{2} - 7\omega^2 C_0 \geq 0. \tag{4.4}$$

We return to $(3.26)_1$, which we write in the form

$$\boldsymbol{\mathcal{L}v} = \frac{1}{\bar{\gamma}}\left[-Re\boldsymbol{v}' + \boldsymbol{b} + \operatorname{div}\left\{\frac{2}{7}\left[N(\boldsymbol{AD}+\boldsymbol{DA}) + N_p(\boldsymbol{A}:\boldsymbol{D})\boldsymbol{I}\right]\right\} - Re(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right] \tag{4.5}$$

We now take the scalar product of (4.5) with $\boldsymbol{\mathcal{L}v}$ in $\mathbb{H}$, and use the Cauchy-Schwarz inequality, to obtain

$$\begin{aligned}
\|\boldsymbol{\mathcal{L}v}\| \quad=\quad & \frac{1}{\gamma}\left[Re^2|\boldsymbol{v}'||\boldsymbol{\mathcal{L}v}| + |\boldsymbol{b}||\boldsymbol{\mathcal{L}v}| + \left(N\|(\boldsymbol{AD}+\boldsymbol{DA})\|\|\boldsymbol{v}\| + \bar{N}_p|\boldsymbol{A}|\,|\boldsymbol{D}|\,\|\boldsymbol{v}\|\right)\right. \\
&\left. + Re\|(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\|\|\boldsymbol{\mathcal{L}v}\|\right].
\end{aligned}$$

Using the Poincaré-Friedrichs and Young inequalities, we deduce from the previous expression that there exist constants $C_0 > 0$ and $C_1 > 0$ such that

$$|\boldsymbol{\mathcal{L}}\boldsymbol{v}|^2 \leq \frac{C_0}{\bar{\gamma}}\left[Re^2|\boldsymbol{v}'|^2 + |\boldsymbol{b}|^2 + \|\boldsymbol{A}\|^2_{\mathbb{H}^2} + C_2 Re^2\|\boldsymbol{v}\|^4\right] \tag{4.6}$$

where

$$\|\boldsymbol{v}\|^2 = \int_\Omega |\nabla \boldsymbol{v}|^2 d\boldsymbol{x}$$

is the Dirichlet norm.

Again we take the scalar product in $\mathbb{H}$ of $(3.26)_1$ with $\boldsymbol{\mathcal{L}}\boldsymbol{v}$, and get

$$\frac{d}{dt}(Re\|\boldsymbol{v}\|^2) + \gamma|\boldsymbol{\mathcal{L}}\boldsymbol{v}|^2 \leq |\boldsymbol{b}||\boldsymbol{\mathcal{L}}\boldsymbol{v}| + Re|(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}|\|\boldsymbol{v}\| + 2N|\boldsymbol{A}\boldsymbol{D}|\|\boldsymbol{v}\| + \bar{N}_p|\boldsymbol{A}||\boldsymbol{D}|\|\boldsymbol{v}\|.$$

Using the Young and Poincaré-Friedrichs inequalities, we deduce from the previous inequality that there exists a constant C_3 such that

$$\frac{d}{dt}(Re\|\boldsymbol{v}\|^2) + \gamma|\boldsymbol{\mathcal{L}}\boldsymbol{v}|^2 \leq \frac{3}{2\gamma}\left(|\boldsymbol{b}|^2 + C_2\|\boldsymbol{A}\|^2_{H^2} + C_3\frac{Re^2}{\gamma^2}\|\boldsymbol{v}\|^6\right). \tag{4.7}$$

Next, we differentiate $(3.26)_1$ with respect to t, and take the scalar product in $\mathbb{H}$ of the resulting equation with $\boldsymbol{v}'$, to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(Re|\boldsymbol{v}'|^2) + \gamma\|\boldsymbol{v}'\|^2 \;\;\leq\;\; & |\boldsymbol{b}'||\boldsymbol{\mathcal{L}}\boldsymbol{v}| + Re\left[\,|(\boldsymbol{v}'\cdot\nabla)\boldsymbol{v}| + |(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}'|\right]|\boldsymbol{\mathcal{L}}\boldsymbol{v}| \\ & + 2N|\boldsymbol{A}\boldsymbol{D}|\|\boldsymbol{v}\| + \bar{N}_p|\boldsymbol{A}||\boldsymbol{D}|\|\boldsymbol{v}\|.\end{aligned}$$

Using the Cauchy-Schwarz and Young inequalities, we deduce that there exists a constant $C = C(N_p, N, \Omega)$ such that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(Re|\boldsymbol{v}'|^2) + \bar{\gamma}\|\boldsymbol{v}'\|^2 \;\;\leq\;\; & \frac{3C^2}{2\gamma}|\boldsymbol{b}'|^2 + \frac{Re^2}{2\epsilon}|\boldsymbol{v}'|^4 + \frac{\epsilon}{2}\|\boldsymbol{v}\|^2_{\mathbb{H}^3} \\ & + \frac{C^2}{\gamma}\left(\|\boldsymbol{A}'\|^4_{H^1} + \|\boldsymbol{v}\|^4 + \|\boldsymbol{A}\|^4 + \|\boldsymbol{A}'\|^4\right).\end{aligned} \tag{4.8}$$

We also differentiate $(3.26)_2$ with respect to t, and take the scalar product in $\mathbb{H}$ of the resulting equation with $(3\gamma/4\omega^2C_0^2)\boldsymbol{A}'$, to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\frac{3\bar{\gamma}We}{4\omega^2C_0^2}|\boldsymbol{A}|^2\right) + \frac{3\bar{\gamma}}{8\omega^2C_0^2}|\boldsymbol{A}'|^2 \;\;\leq\;\; & \frac{\bar{\gamma}}{2}\|\boldsymbol{v}'\|^2 + \frac{We^2}{8\omega^2}\left(\frac{1}{2\epsilon}|\boldsymbol{v}'|^4 + \frac{\epsilon}{2}\|\boldsymbol{A}\|^2_{\mathbb{H}^1}\right) \\ & + \frac{3We^2\bar{\gamma}}{8\epsilon\omega^2}|\boldsymbol{A}'|^4 + \frac{\epsilon}{8}\frac{\bar{\gamma}}{\omega^2C_0^2}\|\boldsymbol{v}\|^2_{\mathbb{H}^3} \\ & + \frac{\bar{\gamma}}{8\omega^2}\left(\frac{\epsilon}{2}\|\boldsymbol{A}\|^4_{\mathbb{H}^2} + \frac{1}{2\epsilon}\|\boldsymbol{v}'\|^4\right).\end{aligned} \tag{4.9}$$

Adding (4.8) and (4.9), we obtain

$$\begin{aligned}
&\frac{d}{dt}\left(Re|\boldsymbol{v}'|^2+\frac{3\bar{\gamma}We}{4\omega^2C_0^2}|\boldsymbol{A}'|^2\right)+\bar{\gamma}\|\boldsymbol{v}'\|^2+\frac{3\bar{\gamma}}{4\omega^2C_0^2}|\boldsymbol{A}'|^2\\
&\leq\frac{3C^2}{\bar{\gamma}}|\boldsymbol{b}'|^2+\frac{1}{\epsilon}\left(\frac{We^2}{8\omega^2}+Re^2\right)|\boldsymbol{v}'|^4+\epsilon\frac{We^2}{8\omega^2}\|\boldsymbol{A}\|^4_{\mathbb{H}^1}\\
&+\epsilon\left(1+\frac{\bar{\gamma}}{4\omega^2C_0^2}\right)\|\boldsymbol{v}\|^3_{H^3}+\frac{3\bar{\gamma}We^2}{4\epsilon\omega^2}\left(\frac{\epsilon}{2}\|\boldsymbol{A}\|^4_{\mathbb{H}^2}+\frac{3\bar{\gamma}We^2}{4\epsilon\omega^2}|\boldsymbol{A}'|^4\right)\\
&+\frac{1}{2\epsilon}\|\boldsymbol{v}'\|^4+\frac{2C^2}{\bar{\gamma}}\left(\|\boldsymbol{A}'\|^4_{H^1}+\|\boldsymbol{v}\|^4+\|\boldsymbol{A}'\|^4+\|\boldsymbol{v}'\|^4\right). \qquad (4.10)
\end{aligned}$$

We multiply inequality (4.3) by $\bar{\gamma}/(12Re^2\omega^2C_0)$, add the resulting inequality to (4.10), make use of (4.2) to estimate $\|\boldsymbol{v}\|^2_{\mathbb{H}^3}$, and use also (4.4), to obtain

$$\begin{aligned}
&\frac{d}{dt}\left(Re|\boldsymbol{v}'|^2+\frac{3\bar{\gamma}We}{4\omega^2C_0^2}|\boldsymbol{A}'|^2+\frac{3\bar{\gamma}We}{4\omega^2C_0^2}\|\boldsymbol{A}\|^2_{\mathbb{H}^2}\right)+\frac{\bar{\gamma}}{2}\|\boldsymbol{v}'\|^2+\frac{3\bar{\gamma}}{4\omega^2C_0^2}|\boldsymbol{A}'|^2+\frac{\bar{\gamma}}{2Re^2}\|\boldsymbol{A}\|^2_{\mathbb{H}^2}\\
&\leq\epsilon C_0\left(1+\frac{\bar{\gamma}}{4\omega^2C_0^2}\right)\left\{Re^2\|\boldsymbol{v}'\|^2+\|\boldsymbol{A}\|^2_{\mathbb{H}^2}+\|\boldsymbol{b}\|^2_{\mathbb{H}^1}+C_2Re^2|\boldsymbol{\mathcal{L}v}|^4\right\}+\frac{3C^2}{\bar{\gamma}}|\boldsymbol{b}'|^2\\
&+\frac{\bar{\gamma}}{2Re^2}\|\boldsymbol{b}\|^2_{\mathbb{H}^1}+\frac{C_0\bar{\gamma}}{2}|\boldsymbol{\mathcal{L}v}|^4+\frac{We^2\bar{\gamma}}{4Re^2C_0\omega^4}\|\boldsymbol{A}\|^4_{\mathbb{H}^2}+\epsilon\frac{We^2}{8\omega^2}\|\boldsymbol{A}\|^4_{\mathbb{H}^1}+\frac{3\bar{\gamma}We^2}{4\epsilon\omega^2}|\boldsymbol{A}'|^4\\
&+\frac{\bar{\gamma}}{4\omega^2}\left(\frac{\epsilon}{2}\|\boldsymbol{A}\|^4_{\mathbb{H}^2}+\frac{1}{\epsilon}\|\boldsymbol{v}'\|^4\right)+\frac{2C^2}{\bar{\gamma}}\left(\|\boldsymbol{A}'\|^4_{\mathbb{H}^1}+\|\boldsymbol{v}\|^4+\|\boldsymbol{A}\|^4+\|\boldsymbol{v}'\|^4\right)\\
&+\frac{1}{\epsilon}\left(\frac{We^2}{8\omega^2}+Re^2\right)|\boldsymbol{v}'|^4. \qquad (4.11)
\end{aligned}$$

We choose $\epsilon=(\bar{\gamma}\omega^2C_0/(Re^2(4\omega^2C_0^2+\bar{\gamma}))$; inequality (4.11) now reads

$$\begin{aligned}
&\frac{d}{dt}\left(Re|\boldsymbol{v}'|^2+\frac{3\bar{\gamma}We}{4\omega^2C_0^2}|\boldsymbol{A}'|^2+\frac{3\bar{\gamma}We}{4\omega^2C_0^2}\|\boldsymbol{A}\|^2_{\mathbb{H}^2}\right)+\frac{\bar{\gamma}}{4}\|\boldsymbol{v}'\|^2+\frac{3\bar{\gamma}}{4\omega^2C_0^2}|\boldsymbol{A}'|^2+\frac{\bar{\gamma}}{4Re^2}\|\boldsymbol{A}\|^2_{\mathbb{H}^2}\\
&\leq\frac{3\bar{\gamma}}{4Re^2}\|\boldsymbol{b}\|^2_{\mathbb{H}^1}+\frac{3C^2}{\bar{\gamma}}|\boldsymbol{b}'|^2+\frac{3C_3\bar{\gamma}}{4}|\boldsymbol{\mathcal{L}v}|^4+\frac{We^2\bar{\gamma}}{4Re^2C_0\omega^4}\|\boldsymbol{A}\|^4_{\mathbb{H}^2}+\frac{C_5\bar{\gamma}We^2}{8Re^2}\|\boldsymbol{A}\|^4_{\mathbb{H}^1}\\
&+\frac{3C_5We^2Re^2}{4\omega^2}|\boldsymbol{A}'|^4+\frac{C_5Re^2}{\bar{\gamma}}|\boldsymbol{v}'|^4+\frac{C_5\bar{\gamma}}{4Re^2}\|\boldsymbol{A}\|^4_{\mathbb{H}^2}+\frac{C_5Re^2}{4\omega^2}\|\boldsymbol{v}'\|^4\\
&+\frac{2C^2}{\bar{\gamma}}\left(\|\boldsymbol{A}'\|^4_{\mathbb{H}^1}+\|\boldsymbol{v}\|^4+\|\boldsymbol{A}\|^4+\|\boldsymbol{v}'\|^4\right), \qquad (4.12)
\end{aligned}$$

where

$$C_5 \quad = \quad \max\left\{\frac{4\omega^2C_0^2+\bar{\gamma}}{4\omega^2C_0},\frac{C_0}{4\omega^2C_0^2+\bar{\gamma}},C_0,\frac{4\omega^2C_0^2+\bar{\gamma}}{\omega^2C_0}\left(\frac{We^2}{8\omega^2}+Re^2\right)\right\}.$$

We multiply (4.7) by $\bar{\gamma}^2/(12C_2Re^2)$, add the resulting inequality to (4.12), and make use of (4.6) to estimate $|\boldsymbol{\mathcal{L}v}|^4$ in (4.12). It follows that there exists $C_6>0$ such that

$$\frac{d}{dt}\left(\frac{\bar{\gamma}^2}{12C_2Re^2}\|\boldsymbol{v}\|^2+Re|\boldsymbol{v}'|^2++\frac{3\bar{\gamma}We}{4\omega^2C_0^2}|\boldsymbol{A}'|^2+\frac{3We\bar{\gamma}}{4\omega^2C_0^2}\|\boldsymbol{A}\|^2_{\mathbb{H}^2}\right)$$

$$
\begin{aligned}
&+\frac{\bar{\gamma}}{8Re^2}\|\boldsymbol{A}\|_{\mathbb{H}^2}^2+\frac{\bar{\gamma}}{4}\|\boldsymbol{v}'\|^2+\frac{3\bar{\gamma}We}{4\omega^2C_0^2}|\boldsymbol{A}'|^2+\frac{\bar{\gamma}^3}{12dRe^2}\|\boldsymbol{v}\|^2\\
&\leq \frac{3\bar{\gamma}}{4Re^2}\|\boldsymbol{b}\|_{\mathbb{H}^1}^2+\frac{3\bar{\gamma}}{8C_2Re^2}|\boldsymbol{b}|^2+\frac{3C^2}{\bar{\gamma}}|\boldsymbol{b}'|^2+\frac{3C_6}{4\bar{\gamma}\bar{\gamma}}|\boldsymbol{b}|^4+\frac{3C_5We^2Re^2}{4\omega^2}|\boldsymbol{A}'|^4\\
&+C_7\left\{\frac{\bar{\gamma}}{Re^2}\|\boldsymbol{A}\|_{\mathbb{H}^1}^4+\frac{1}{\bar{\gamma}}\|\boldsymbol{v}\|^4+\frac{Re^2}{\bar{\gamma}}|\boldsymbol{v}'|^4+\frac{\bar{\gamma}}{Re^2}\|\boldsymbol{A}\|_{\mathbb{H}^2}^4+\frac{Re^2}{\bar{\gamma}}\|\boldsymbol{v}'\|^4\right\}\\
&+\frac{2C^2}{\bar{\gamma}}\|\boldsymbol{A}\|^4+\frac{2C^2}{\bar{\gamma}}\|\boldsymbol{A}'\|_{\mathbb{H}^1}^4+\|\boldsymbol{v}\|^6+C_7\frac{3Re^4}{4\bar{\gamma}}\|\boldsymbol{v}\|^8, \qquad (4.13)
\end{aligned}
$$

with

$$
\begin{aligned}
C_7 \;=\; \max&\left\{\frac{C_5We^2}{8}+\frac{3C_6Re^2}{\bar{\gamma}^2}, 2C^2+\frac{C_3}{8C_2}, 3c_6Re^2+C^5, \frac{We^2}{4C_0\omega^4}+\frac{C_5}{4}\frac{3C_3C_6}{\bar{\gamma}^2},\right.\\
&\left.\frac{C_5\bar{\gamma}}{4\omega^2}+\frac{2C^2}{Re^2}C_6C_2^2\right\}.
\end{aligned}
$$

We now state an important result which will help us to show the global existence of regular solution.

Lemma 5 [9] *Let f be a non negative, absolutely continuous function satisfying the inequality*

$$
f'+kf\leq \alpha\left(f^2+f^3+f^4+f^6+f^{2m}\right)+\beta \qquad (4.14)
$$

where m≥ 2, $k>0$, $\alpha>0$ and $\beta\geq 0$ are some constants. Let $M_0>0$, be the unique solution of $M^{2m-1}+M^5+M^3+M^2+M-\frac{k}{2\alpha}=0$, and $0<M<M_0$. If f(0)$\leq$M, and $\beta\leq\frac{kM}{3}$, then f(t) is bounded by M for all t>0.

Corollary 1 *Under the hypotheses of Lemma 5, f(t)$\leq$M $\forall t\geq 0$, and inequality (4.14) implies that*

$$
f'+kf\leq \eta f^2+\beta, \qquad (4.15)
$$

where

$$
\eta=\alpha\left(1+M+M^2+M^4+M^{2m-1}\right)>0.
$$

Global existence and stability of regular solution: linear closure. Let

$$
f(t) \;=\; \frac{\bar{\gamma}^2}{12C_2Re^2}\|\boldsymbol{v}(t)\|^2+Re|\boldsymbol{v}'(t)|+\frac{3\bar{\gamma}We}{4\omega^2C_0^2}|\boldsymbol{A}'(t)|^2+\frac{3\bar{\gamma}We}{4\omega^2C_0^2}\|\boldsymbol{A}(t)\|_{\mathbb{H}^2}^2. \qquad (4.16)
$$

Then inequality (4.13) takes the form:

$$f'(t) + kf(t) \leq \alpha \big(f^2(t) + f^3(t) + f^4(t) \big) + \beta, \tag{4.17}$$

where $k > 0$, $\alpha > 0$ and $\beta \geq 0$ are constants depending on the data.

Theorem 5 (Global existence for the linear closure)
Let $\partial\Omega \in C^4$. If $\bar{\gamma} > 0$, there exists ω_0 satisfying (4.4) and depending on the data, such that, if $\omega_0 < \omega$ and $\boldsymbol{v}_0 \in D(\boldsymbol{\mathcal{L}})$, $\boldsymbol{A}_0 \in \mathbb{H}^2$, $\boldsymbol{b} \in \mathbb{L}^\infty(\mathbb{R}_+, \mathbb{H}^2)$, and $\boldsymbol{b}' \in \mathbb{L}^\infty(\mathbb{R}_+, \mathbb{H}^1)$ are small enough in their spaces, then the problem (3.26) admits a unique solution ($\boldsymbol{v}$,$\boldsymbol{A}$) defined for all times $t > 0$, and

$$\begin{aligned}
&\boldsymbol{v} \in C_b(\mathbb{R}_+, D(\boldsymbol{\mathcal{L}})) \cap \mathbb{L}^2_{loc}(\mathbb{R}_+, \mathbb{H}^3)\\
&\boldsymbol{v}' \in C_b(\mathbb{R}_+, \mathbb{H}) \cap \mathbb{L}^2_{loc}(\mathbb{R}_+, \mathbb{V})\\
&\boldsymbol{A} \in C_b(\mathbb{R}_+, \mathbb{H}^2) \cap \mathbb{X}, \qquad \boldsymbol{A}' \in C_b(\mathbb{R}_+, \mathbb{H}^1).
\end{aligned}$$

Proof. Step 1. From (4.13), we see that the local solution obtained in Theorem 1 satisfies inequality (4.17) with

$$\beta \;=\; \frac{3\bar{\gamma}}{4Re^2}\|\boldsymbol{b}\|^2_{L^\infty(0,T;\mathbb{H}^1)} + \frac{\bar{\gamma}}{8C_2 Re^2}|\boldsymbol{b}|^2 + \frac{3C^2}{\bar{\gamma}}|\boldsymbol{b}'|^2 + \frac{3C_6}{4\bar{\gamma}\bar{\gamma}}|\boldsymbol{b}|^4.$$

By Lemma 5, there exists a constant M_0, depending on the data, such that if $f(0) \leq M \leq M_0$ and $\beta \leq \frac{kM}{3}$, then $f(t)$ is bounded for all $t \in \mathbb{R}_+$. Observe also that $f(0) \leq M$ if $\boldsymbol{v}_0$, $\boldsymbol{A}_0$ and $\boldsymbol{b}$ are small in their respective spaces. Therefore from the hypotheses, if $\boldsymbol{b} \in \mathbb{L}^\infty(\mathbb{R}_+; \mathbb{H}^2)$, $\boldsymbol{b}' \in \mathbb{L}^2(\mathbb{R}_+, \mathbb{H}^1)$, we deduce that

$$\begin{aligned}
&\boldsymbol{v} \in \mathbb{L}^\infty(\mathbb{R}_+, \mathbb{V}) \cap \mathbb{L}^2_{loc}(\mathbb{R}_+, D(\boldsymbol{\mathcal{L}})),\\
&\boldsymbol{v}' \in \mathbb{L}^\infty(\mathbb{R}_+, \mathbb{H}) \cap \mathbb{L}^2_{loc}(\mathbb{R}_+, \mathbb{V}),\\
&\boldsymbol{A} \in \mathbb{L}^\infty(\mathbb{R}_+, \mathbb{H}^2), \qquad \boldsymbol{A}' \in \mathbb{L}^\infty(\mathbb{R}_+, \mathbb{L}^2).
\end{aligned} \tag{4.18}$$

Step 2. From inequality (4.6) we deduce that $\boldsymbol{v} \in \mathbb{L}^\infty(\mathbb{R}_+; D(\boldsymbol{\mathcal{L}}))$ and from (4.5), that $\boldsymbol{v} \in \mathbb{L}^2_{loc}(\mathbb{R}_+, \mathbb{H}^3)$. In the same way (4.18) implies that $\boldsymbol{v} \in C_b(\mathbb{R}_+; D(\boldsymbol{\mathcal{L}}))$, and $(3.26)_1$ implies that $\boldsymbol{v}' \in C_b(\mathbb{R}_+; \mathbb{H})$.

We write $(3.26)_2$ in the form

$$\begin{aligned} We\{\boldsymbol{A}' + (\boldsymbol{v}\cdot\nabla)\boldsymbol{A} + \boldsymbol{A}\} &= -We\big\{\boldsymbol{A}\boldsymbol{W} - \boldsymbol{W}\boldsymbol{A} - 3\bar{\lambda}(\boldsymbol{A}\boldsymbol{D} + \boldsymbol{D}\boldsymbol{A})\big\} \\ &\quad +2\omega\boldsymbol{D} - 2\bar{\lambda}We(\boldsymbol{A}:\boldsymbol{D})\boldsymbol{I}. \end{aligned} \tag{4.19}$$

The right hand side of (4.19) has its first and third terms belonging to $\mathbb{L}^2_{loc}(\mathbb{R}_+, \mathbb{H}^2)$, so that $\boldsymbol{A} \in C_b(\mathbb{R}_+, \mathbb{H}^3)$. This together with (4.18) implies that the right-hand side of (4.19) belongs to $C_b(\mathbb{R}_+, \mathbb{H}^1)$ and that $\boldsymbol{A}' \in C_b(\mathbb{R}, \mathbb{H}^1)$, because we also have $\boldsymbol{A}'(0) \in \mathbb{H}^1$ (from the hypotheses of the theorem). □

Corollary 2 (STABILITY OF SOLUTION IN THE ABSENCE OF BODY FORCE)
Under the hypotheses of Theorem 5, and in the absence of body force, the solution $(\boldsymbol{v},\boldsymbol{A})$ *obtained in Theorem 5 is exponentially stable.*

Proof. We choose ω_0 such that it satisfies (4.4). Therefore the solution $(\boldsymbol{v},\boldsymbol{A})$ satisfies (4.13), (4.17), (4.14) and consequently (4.15), with $\beta = 0$. Therefore, from (4.15) we deduce that

$$f(t) \leq \frac{f(0)e^{-kt}}{1 - \frac{\eta}{k}f(0)}, \tag{4.20}$$

with $1 - \dfrac{\eta}{k}f \geq 0$. In particular, we use (4.16), which with (4.20) implies that

$$\|\boldsymbol{v}\|^2_{\mathbb{L}^2} + \|\boldsymbol{A}\|^2_{\mathbb{L}^2} \leq Ke^{-kt}, \tag{4.21}$$

where K is a positive constant depending on the data. □

4.2 Quadratic closure approximation

In this Section, we show, as in the previous Section, that the unique solution obtained in Theorems 3 and 4 is defined on $\mathbb{R}_+$, if the data are small enough. As in the case of the linear closure, a priori estimates may be derived for the quadratic closure, and we find that $\boldsymbol{v}$ and $\boldsymbol{A}$ satisfy the inequality (see [11] for more details)

$$\frac{d}{dt}\left(\frac{\bar{\gamma}^2}{92C_2Re}\|\boldsymbol{v}\|^2 + Re|\boldsymbol{v}'|^2 + \frac{\bar{\gamma}}{80C^2\lambda^2\eta}|\boldsymbol{A}'|^2 + \frac{\bar{\gamma}}{12C_0\lambda^2\eta Re^2}\|\boldsymbol{A}\|^2_{\mathbb{H}^2}\right)$$

$$
\begin{aligned}
&+\frac{\bar{\gamma}}{48Re^2}\|\boldsymbol{A}\|^2_{\mathbb{H}^2}+\frac{\bar{\gamma}}{24}\|\boldsymbol{v}'\|^2+\frac{\bar{\gamma}}{160C^2\lambda^2\eta^2}|\boldsymbol{A}'|^2+\frac{\bar{\gamma}^3}{96C_2Re^2d}\|\boldsymbol{v}\|^2\\
&\leq \frac{4C^2}{\bar{\gamma}}|\boldsymbol{b}'|^2+\frac{7\bar{\gamma}}{8Re^2}\|\boldsymbol{b}\|^2_{\mathbb{H}^1}+\frac{C_0\bar{\gamma}}{48C_2Re^2}|\boldsymbol{b}|^2+C_5\lambda^2\eta\|\boldsymbol{b}\|^4_{\mathbb{H}^1}+\frac{7C_6}{24\bar{\gamma}}|\boldsymbol{b}|^4+\frac{C_6Re^4}{\bar{\gamma}^4}|\boldsymbol{b}|^8\\
&+\frac{C_7}{\bar{\gamma}}\left(\|\boldsymbol{v}\|^4+\|\boldsymbol{A}'\|^4+Re^2|\boldsymbol{v}'|^4\right)+C_7\frac{\bar{\gamma}}{12Re^2}\left(\|\boldsymbol{v}\|^4_{\mathbb{H}^2}+12\|\boldsymbol{v}\|^4_{\mathbb{H}^1}\right)\\
&+C_5\frac{\bar{\gamma}}{4}\left(\|\boldsymbol{v}'\|^4+\frac{2}{\lambda^2\eta^2}|\boldsymbol{A}'|^4\right)+\frac{C_7}{\bar{\gamma}}\left(\|\boldsymbol{A}\|^8_{\mathbb{H}^1}+\|\boldsymbol{A}'\|^8+\|\boldsymbol{v}\|^8\right)\\
&+Re^4\frac{C_7}{\bar{\gamma}^4}\left(Re^8|\boldsymbol{v}'|^8+\|\boldsymbol{A}\|^8_{\mathbb{H}^2}+\|\boldsymbol{A}\|^8\right)+\frac{C_1\bar{\gamma}}{12C_0\lambda^4\eta^2Re^2}\|\boldsymbol{A}\|^6_{\mathbb{H}^2}\\
&+Re^4\frac{C_7}{\bar{\gamma}^4}\left(\|\boldsymbol{A}\|^{16}_{\mathbb{H}^1}+C_2Re^2\|\boldsymbol{A}\|^{16}\right), \qquad (4.22)
\end{aligned}
$$

where $C, C_0, C_1, C_2, C_5, C_6, C_7$ and d are positive constants depending on the data.

We state without proof the results concerning global existence and stability of solutions, for the quadratic closure.

Theorem 6 (GLOBAL EXISTENCE FOR THE QUADRATIC CLOSURE)
Let $\partial\Omega \in C^4$. There exists η_0 depending on Ω, N_s and on the data, such that, if $\eta_0 < \eta$ and $\boldsymbol{v}_0 \in D(\boldsymbol{\mathcal{L}})$, $\boldsymbol{A}_0 \in \mathbb{H}^2$, $\boldsymbol{b} \in \mathbb{L}^\infty(\mathbb{R}_+, \mathbb{H}^2)$, and $\boldsymbol{b}' \in \mathbb{L}^\infty(\mathbb{R}_+, \mathbb{H}^1)$ are small enough in their spaces, then the problem (3.60) admits a unique solution ($\boldsymbol{v}$,$\boldsymbol{A}$) defined for all times t, and

$$
\begin{aligned}
&\boldsymbol{v} \in C_b(\mathbb{R}_+, D(\boldsymbol{\mathcal{L}})) \cap \mathbb{L}^2_{loc}(\mathbb{R}_+, \mathbb{H}^3)\\
&\boldsymbol{v}' \in C_b(\mathbb{R}_+, \mathbb{H}) \cap \mathbb{L}^2_{loc}(\mathbb{R}_+, \mathbb{V})\\
&\boldsymbol{A} \in C_b(\mathbb{R}_+, \mathbb{H}^2) \cap \mathbb{X}, \qquad \boldsymbol{A}' \in C_b(\mathbb{R}_+, \mathbb{H}^1).
\end{aligned}
$$

Corollary 3 (STABILITY OF SOLUTION AROUND ZERO FOR THE QUADRATIC CLOSURE)
Under the hypotheses of Theorem 6, we assume also that $\boldsymbol{b}$=$\boldsymbol{0}$. Then the solution ($\boldsymbol{v}$,$\boldsymbol{A}$) obtained in Theorem 6 is exponentially stable.

5 Solution to the steady problem

We turn now to the steady versions of problems (3.26) and (3.60). These are, for the linear closure,

$$\left.\begin{aligned} \mathcal{L}\boldsymbol{v} = \boldsymbol{b} - Re(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} \\ +\mathrm{div}\,\{N(\boldsymbol{AD}+\boldsymbol{DA}) + \bar{N}_p(\boldsymbol{A}:\boldsymbol{D})\boldsymbol{I}\} \\ \\ We\{\boldsymbol{A}' + (\boldsymbol{v}\cdot\nabla)\boldsymbol{A} + \boldsymbol{AW} - \boldsymbol{WA} - 3\bar{\lambda}(\boldsymbol{AD}+\boldsymbol{DA})\} + \boldsymbol{A} \\ = 2\omega\boldsymbol{D} - 2\bar{\lambda}We(\boldsymbol{A}:\boldsymbol{D})\boldsymbol{I} \\ \\ \boldsymbol{v}\in\mathbb{V}, \quad \boldsymbol{A}\in\mathbb{H}^2(\Omega)\cap\mathbb{X} \end{aligned}\right\} \tag{5.1}$$

and for the quadratic closure,

$$\left.\begin{aligned} &\bar{\gamma}\mathcal{L}\boldsymbol{v} = 2\mathrm{div}\,\left[N_p(\boldsymbol{A}:\boldsymbol{D})(\boldsymbol{A}+\tfrac{1}{d}\boldsymbol{I}) + N_s(\boldsymbol{AD}+\boldsymbol{DA})\right] \\ &+\boldsymbol{b} - Re(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}, \\ &\eta\,\{(\boldsymbol{v}\cdot\nabla)\boldsymbol{A} + (\boldsymbol{AW}-\boldsymbol{WA}) - \lambda(\boldsymbol{AD}+\boldsymbol{DA})\} + \boldsymbol{A} \\ &= 2\lambda\eta\boldsymbol{D} - 2\lambda\eta(\boldsymbol{A}:\boldsymbol{D})(\boldsymbol{A}+\tfrac{1}{d}\boldsymbol{I}), \\ \\ &\boldsymbol{v}\in\mathbb{V}, \qquad \boldsymbol{A}\in\mathbb{H}^2(\Omega)\cap\mathbb{X} \end{aligned}\right\} \tag{5.2}$$

5.1 Linear closure

In this section, we show that the problem (5.1) has a solution which is unique if the data are small enough.

Linearized problems. Let us consider the linearized problem of (5.1), viz.

$$\left.\begin{aligned} &\gamma\mathcal{L}\boldsymbol{v} = \boldsymbol{F} \\ &\mathrm{div}\,\boldsymbol{v} = 0 \\ &\boldsymbol{v}\in\mathbb{V} \end{aligned}\right\} \tag{5.3}$$

where $\boldsymbol{F}$ is a given body force.

And for a given velocity field $\overline{\boldsymbol{v}}$, we consider the problem

$$\left.\begin{array}{r} We\{(\overline{\boldsymbol{v}}\cdot\nabla)\boldsymbol{A}+\boldsymbol{A}\overline{\boldsymbol{W}}-\overline{\boldsymbol{W}}\boldsymbol{A}-3\bar{\lambda}(\boldsymbol{A}\overline{\boldsymbol{D}}+\overline{\boldsymbol{D}}\boldsymbol{A})\}+\boldsymbol{A} \\ \\ =2\omega\overline{\boldsymbol{D}}-2\bar{\lambda}We(\boldsymbol{A}:\overline{\boldsymbol{D}})\boldsymbol{I} \\ \\ \boldsymbol{A}\in\mathbb{X} \end{array}\right\} \tag{5.4}$$

Lemma 6 *Assume that $\Gamma\in C^3$, and $\boldsymbol{F}\in\mathbb{H}^2(\Omega)$. If $\gamma>0$, then the Stokes problem (5.3) admits a unique solution $\boldsymbol{v}\in\mathbb{H}^3(\Omega)$ and there exists $C_0(\gamma,\Omega)$, such that*

$$\|\boldsymbol{v}\|_{\mathbb{H}^3(\Omega)}\le C_0\|\boldsymbol{F}\|_{\mathbb{H}^1(\Omega)}. \tag{5.5}$$

Proof (See [14], page 33.)

Lemma 7 *Assume that $\Gamma\in C^1$, and that $\overline{\boldsymbol{v}}\in\mathbb{H}^3(\Omega)\cap D(\boldsymbol{\mathcal{L}})$, with $\|\overline{\boldsymbol{v}}\|_{\mathbb{H}^3}\le CD$. Then the problem (5.4) admits a unique solution $\boldsymbol{A}\in\mathbb{H}^2(\Omega)$. Futhermore, there exists a constant $c(\Omega,\omega,We)$ such that if $D<c$ then*

$$\|\boldsymbol{A}\|_{\mathbb{H}^2}\le D. \tag{5.6}$$

Proof

Estimate (5.6)

We proceed exactly as in Lemma 3 to obtain the analogue of (3.14), in the form

$$\begin{aligned}\|\boldsymbol{A}\|^2_{\mathbb{H}^2} &= 2\omega(\overline{\boldsymbol{D}},\boldsymbol{A})_{\mathbb{H}^2}+We(\overline{\boldsymbol{W}}\boldsymbol{A}-\boldsymbol{A}\overline{\boldsymbol{W}},\boldsymbol{A})_{\mathbb{H}^2}\\ &\quad -We\int_\Omega\{\bar{v}_{k,l}\left[\boldsymbol{A}_{,k}:\boldsymbol{A}_{,l}+2\boldsymbol{A}_{,km}:\boldsymbol{A}_{,lm}\right]+\bar{v}_{k,lm}\boldsymbol{A}_{,k}:\boldsymbol{A}_{,lm}\}d\boldsymbol{x}\\ &\quad -2\bar{\lambda}We\left(((\boldsymbol{A}:\overline{\boldsymbol{D}})\boldsymbol{I},\boldsymbol{A})_{\mathbb{H}^2}+3\bar{\lambda}We(\boldsymbol{A}\overline{\boldsymbol{D}}+\overline{\boldsymbol{D}}\boldsymbol{A},\boldsymbol{A})_{\mathbb{H}^2}\right).\end{aligned} \tag{5.7}$$

The right-hand side of (5.7) may be estimated as in the case of (3.14), and we deduce that

$$\begin{aligned}\|\boldsymbol{A}\|_{\mathbb{H}^2} &\le C_0(\|\overline{\boldsymbol{v}}\|_{\mathbb{H}^3}+2We\|\overline{\boldsymbol{v}}\|_{\mathbb{H}^3}\|\boldsymbol{A}\|_{\mathbb{H}^2}).\\ &\le C_1(CD+CD\|\boldsymbol{A}\|_{\mathbb{H}^2}).\end{aligned}$$

We impose $C_1C(1+D)<1$, to obtain (5.6).

Theorem 7 *Assume that* $\Gamma \in C^3, \boldsymbol{b} \in \mathbb{H}^2(\Omega)$. *If* $\gamma > 0$, *then there exists a positive constant* $c = c(\Omega, \omega, We)$ *such that* $(\boldsymbol{v}, \boldsymbol{A}, p) \in (\mathbb{H}^3(\Omega) \cap \mathbb{V}) \times (\mathbb{H}^2(\Omega) \cap \mathbb{X} \times \mathbb{H}^2(\Omega))$ *is the unique solution to Problem (5.1), with*

$$||\boldsymbol{v}||_{\mathbb{H}^3} + ||\boldsymbol{A}||_{\mathbb{H}^2} \leq c. \tag{5.8}$$

Proof. The proof draws substantially from [7].

Step 1. For $B > 0$, we define the set

$$R = \{\ (\overline{\boldsymbol{v}}, \overline{\boldsymbol{A}}), \overline{\boldsymbol{v}} \in \mathbb{H}^3(\Omega); \overline{\boldsymbol{A}} \in \mathbb{H}^2(\Omega); \quad ||\overline{\boldsymbol{v}}||_{\mathbb{H}^3} \leq B, \quad ||\overline{\boldsymbol{A}}||_{\mathbb{H}^2} \leq B\,\}\,. \tag{5.9}$$

Next, we consider the mapping

$$\begin{aligned} \phi : R &\longrightarrow X = \mathbb{V} \times \mathbb{H}^1(\Omega), \\ (\overline{\boldsymbol{v}}, \overline{\boldsymbol{A}}) &\longmapsto (\boldsymbol{v}, \boldsymbol{A}), \end{aligned}$$

where $\boldsymbol{A}$ and $\boldsymbol{v}$ are the unique solutions of (5.3) and (5.4) respectively, with

$$\boldsymbol{F} = -Re(\overline{\boldsymbol{v}} \cdot \nabla)\overline{\boldsymbol{v}} + \boldsymbol{b} + \operatorname{div}\left\{N(\overline{\boldsymbol{A}}\ \overline{\boldsymbol{D}} + \overline{\boldsymbol{D}}\ \overline{\boldsymbol{A}}) + \bar{N}_p(\overline{\boldsymbol{A}} : \overline{\boldsymbol{D}})\boldsymbol{I}\right\}.$$

From Lemmas 6 and 7, we see that ϕ is well-defined and, if D is suitably chosen, $\phi(R) \subset R$. As in the proof of Theorem 1, one can show that ϕ is a fixed point $(\boldsymbol{v}, \boldsymbol{A})$, and is obviouly a solution to problem (5.2). The estimate (5.8) is deduced immediately if we choose $c = D^2$. For uniqueness, as usual, we take the difference of two solutions $(\boldsymbol{v}_1, \boldsymbol{A}_1, p_1)$ and $(\boldsymbol{v}_2, \boldsymbol{A}_2, p_2)$, corresponding to the same data. Set $\boldsymbol{v}$=$\boldsymbol{v}_1 - \boldsymbol{v}_2$ and $\boldsymbol{A}$=$\boldsymbol{A}_1 - \boldsymbol{A}_2$. We make use of (3.48) and (3.49), and find that the functions $\boldsymbol{v}$ and $\boldsymbol{A}$ satisfy the estimates

$$\begin{aligned} ||\boldsymbol{v}||^2_{\mathbb{H}^2} &\leq \frac{C_0}{\gamma}\{N||\boldsymbol{A}_1||_{\mathbb{L}^2}||\boldsymbol{v}||^2_{\mathbb{H}^2} + ||\boldsymbol{A}||_{\mathbb{L}}||\boldsymbol{v}_2||_{\mathbb{H}^2}\} && (5.10) \\ ||\boldsymbol{A}||^2_{\mathbb{L}^2} &\leq 2C_0\omega||\boldsymbol{A}||_{\mathbb{L}^2}|\boldsymbol{v}||_{\mathbb{H}^2} + WeC_0\{||\boldsymbol{A}_1||_{\mathbb{H}^2}||\boldsymbol{v}||_{\mathbb{H}^2}||\boldsymbol{A}||_{\mathbb{L}^2} + ||\boldsymbol{v}_2||_{\mathbb{H}^3}||\boldsymbol{A}||^2_{\mathbb{L}^2}\} \\ &\leq \frac{1}{2}C_1\left\{(C_0 + C_2||\boldsymbol{A}_1||_{\mathbb{H}^2})||\boldsymbol{v}||^2_{\mathbb{H}^2} + \frac{1}{2}||\boldsymbol{A}||^2_{\mathbb{L}^2} + We||\boldsymbol{v}_2||^2_{\mathbb{H}^3}||\boldsymbol{A}||^2_{\mathbb{L}^2}\right\} && (5.11) \end{aligned}$$

where $C_1 = C_1(We, \Omega, \gamma)$, $C_0 = C_0(\omega, \Omega)$.

Next, we add (5.10) and (5.11) to obtain

$$\begin{aligned} ||\boldsymbol{v}||^2_{\mathbb{H}^2} + ||\boldsymbol{A}||^2_{\mathbb{L}^2} &\leq C_2(||\boldsymbol{A}_1||_{\mathbb{H}^2} + C_3)||\boldsymbol{v}||^2_{\mathbb{H}^2} + C_4(||\boldsymbol{v}_2||_{\mathbb{H}^2} + \tfrac{1}{2} + We||\boldsymbol{v}_2||^2_{\mathbb{H}^3}||\boldsymbol{A}||^2_{\mathbb{L}^2} \\ &\leq C_5(C + C_0)||\boldsymbol{v}||^2_{\mathbb{H}^2} + C_5(c + \tfrac{1}{2})||\boldsymbol{A}||^2_{\mathbb{L}^2} \\ &\leq \alpha||\boldsymbol{v}||^2_{\mathbb{H}^2} + \boldsymbol{\eta} && (5.12) \end{aligned}$$

where $\boldsymbol{\eta} = C_5(c + \frac{1}{2})\|\boldsymbol{A}\|^2_{\mathbb{L}^2}$, and α and β depend on $We, \Omega, \mu_I/\mu$ and ω. Uniqueness is established if α and β are small enough. $\square$

5.2 Quadratic Closure

We state the result on existence and the uniqueness of a solution to the problem (5.2). The proof is obtained by modifying that for the unsteady problem in the same way as for the linear closure.

Theorem 8 *Assume that* $\Gamma \in C^3, \boldsymbol{b} \in \mathbb{H}^2(\Omega)$. *There exists a positive constant* $c = c(\Omega, \omega, We)$ *such that* $(\boldsymbol{v}, \boldsymbol{A}, p) \in (\mathbb{H}^3(\Omega) \cap \mathbb{V}) \times (\mathbb{H}^2(\Omega) \cap \mathbb{X} \times \mathbb{H}^2(\Omega))$ *is the solution to Problem (5.2), with*

$$\|\boldsymbol{v}\|_{\mathbb{H}^3} + \|\boldsymbol{A}\|_{\mathbb{H}^2} \leq c. \tag{5.13}$$

This solution is unique if the data is small enough.

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