

# ON THE STABILIZATION OF THE RECTANGULAR FOUR-NODE QUADRILATERAL ELEMENT

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## SUMMARY

The standard bilinear displacement field of the plane linear elastic rectangular four-node quadrilateral element is enhanced by incompatible modes. The resulting gradient operators are separated into constant and linear parts corresponding to underintegration and stabilization of the element stiffness matrix. Minimization of potential energy is used to generate exact analytical expressions for the hourglass stabilization of the rectangle. The stabilized element is shown to coincide with the element obtained by the mixed assumed strain method.

# 1 Introduction

The standard displacement based four-node quadrilateral finite element is not suitable for the analysis of problems with coarse meshes or problems involving incompressible materials. Throughout the development of the finite element method numerous efforts have been made to describe and to avoid this major disadvantage of the quadrilateral.

In the special case of a rectangular geometry several different approaches lead to different results for the element stiffness matrix.<sup>1-6</sup> Due to the convergence of the finite element solutions when increasing the number of elements, these different formulations may still be used for the analysis. Some of the suggested procedures all yield exactly the same stiffness matrix.<sup>7</sup> This matrix associates each deformation mode of the rectangle with the corresponding correct stiffness. Therefore this one particular matrix provides for accurate displacement solutions at the nodes of the rectangle.

As a matter of course, it is desirable to achieve such accuracy even for meshes of quadrilaterals with arbitrary element geometry. Here, major accomplishments have been made with element formulations based on mixed methods.<sup>8-11</sup>

One of the procedures to overcome the difficulties associated with the standard element formulation focuses on the ‘underintegration’ of the quadrilateral by using a 1-point Gauss quadrature rule for an efficient evaluation of the element stiffness matrix.<sup>12,13</sup> To provide for stability of the solution a subsequent treatment of the stiffness matrix becomes inevitable. This is often referred to as ‘hourglass control’.<sup>14-16</sup>

An eigensystem analysis was used to show that the numerical accuracy of the element is almost independent of the value of the hourglass stabilization parameter over a wide range of values.<sup>17</sup> A rather complicated closed form solution for the stabilization has been given in the context of an assumed stress hybrid element.<sup>18,19</sup> Other recent approaches suggest yet different stabilization procedures for the quadrilateral, depending on the type of problem to be analyzed.<sup>3-5</sup>

In this paper a simple, displacement based, closed form solution is developed for the stabilization of the rectangular quadrilateral. The resulting element yields accurate displacement solutions at the element nodes. The formulation includes the cases of plane stress and plane strain, as well as the analysis of incompressible materials.

The described element stiffness matrix turns out to be identical with those stiffness matrices of rectangles obtained from assumed stress and assumed strain mixed methods.<sup>8,9</sup> For the assumed strain method<sup>9,20</sup> the identity is derived analytically. The identity can also be observed from a comparison of numerical eigensystem analyses of unsupported rectangles.

## 2 Notation

The mapping of the rectangular element from the physical space into the master element space is shown in Figure 1. The transformation of co-ordinates and derivatives is

$$x = \frac{a}{2} \xi \quad y = \frac{b}{2} \eta \quad (1)$$

$$\frac{\partial \xi}{\partial x} = \frac{2}{a} \quad \frac{\partial \eta}{\partial x} = 0 \quad \frac{\partial \xi}{\partial y} = 0 \quad \frac{\partial \eta}{\partial y} = \frac{2}{b} \quad (2)$$

The determinant of the Jacobian of the mapping becomes

$$J = \frac{1}{4} a b \quad (3)$$

The standard displacement field  $u$  in the  $x$ -direction and  $v$  in the  $y$ -direction is written in terms of the vector of the bilinear element shape functions,  $\mathbf{n}$ , and the vector of nodal displacements,  $\mathbf{d}$

$$\mathbf{d} = \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} \quad u = \mathbf{n}^T \mathbf{u} \quad v = \mathbf{n}^T \mathbf{v} \quad (4)$$

The shape functions can be separated into constant, linear, and bilinear parts by setting

$$\mathbf{n} = \frac{1}{4} \begin{pmatrix} (1 - \xi)(1 - \eta) \\ (1 + \xi)(1 - \eta) \\ (1 + \xi)(1 + \eta) \\ (1 - \xi)(1 + \eta) \end{pmatrix} = \frac{1}{2} \left( \mathbf{r} + \mathbf{g}_\xi \xi + \mathbf{g}_\eta \eta + \mathbf{h} \xi \eta \right) \quad (5)$$

with

$$\mathbf{r} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad \mathbf{g}_\xi = \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix} \quad \mathbf{g}_\eta = \frac{1}{2} \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix} \quad \mathbf{h} = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \quad (6)$$

where  $\mathbf{r}$ ,  $\mathbf{g}$ , and  $\mathbf{h}$  stand for ‘rigid body’, ‘constant gradient’, and ‘hourglass’, respectively. The derivatives of the element shape functions with respect to  $x$  and  $y$  are

$$\begin{aligned} \frac{\partial \mathbf{n}}{\partial x} &= \frac{1}{a} \left( \mathbf{g}_\xi + \mathbf{h} \eta \right) \\ \frac{\partial \mathbf{n}}{\partial y} &= \frac{1}{b} \left( \mathbf{g}_\eta + \mathbf{h} \xi \right) \end{aligned} \quad (7)$$

The matrix of material stiffnesses,  $\mathbf{E}$ , is given by

$$\mathbf{E} = \begin{pmatrix} E_1 & E_2 & 0 \\ E_2 & E_1 & 0 \\ 0 & 0 & G \end{pmatrix} \quad (8)$$

For plane stress

$$E_1 = \frac{E}{1 - \nu^2} \quad E_2 = \nu E_1 \quad (9)$$

while for plane strain

$$E_1 = \frac{E(1 - \nu)}{(1 + \nu)(1 - 2\nu)} \quad E_2 = \frac{\nu E_1}{1 - \nu} \quad (10)$$

with Young's modulus  $E$ , Poisson's ratio  $\nu$ , and the shear modulus  $G$ .

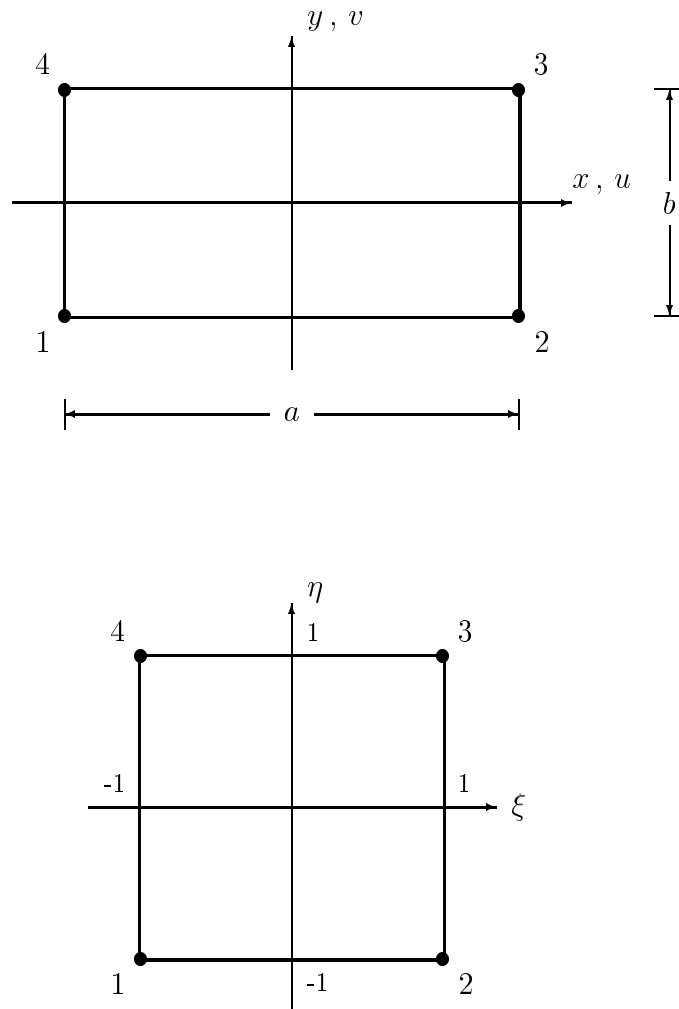


Figure 1: Mapping of the rectangular 4-node quadrilateral element

### 3 Variational principle

The element stiffness matrix will be obtained from the minimization of the potential energy of the linear elastic body

$$\Pi = \frac{1}{2} \int \boldsymbol{\epsilon}^T \mathbf{E} \boldsymbol{\epsilon} dV - \Pi_L \rightarrow \text{Min.} \quad (11)$$

where  $\boldsymbol{\epsilon}$  is the strain field,  $\Pi_L$  is the potential of the external loads, and integration is over the volume  $V$ .

The standard displacement field is now enhanced by non-conforming functions<sup>21,22</sup> such that completeness of polynomials up to second order is given in  $\xi$  and  $\eta$ :

$$\begin{aligned} u &= \mathbf{n}^T \mathbf{u} + \varphi_1 (1 - \xi^2) + \varphi_2 (1 - \eta^2) \\ v &= \mathbf{n}^T \mathbf{v} + \varphi_3 (1 - \xi^2) + \varphi_4 (1 - \eta^2) \end{aligned} \quad (12)$$

The coefficients

$$\boldsymbol{\varphi}^T = \left( \varphi_1 \quad \varphi_2 \quad \varphi_3 \quad \varphi_4 \right) \quad (13)$$

are a priori unknown. The element strains

$$\begin{aligned} \epsilon_x &= \frac{\partial u}{\partial x} = \frac{\partial \mathbf{n}^T}{\partial x} \mathbf{u} - \frac{4}{a} \varphi_1 \xi \\ \epsilon_y &= \frac{\partial v}{\partial y} = \frac{\partial \mathbf{n}^T}{\partial y} \mathbf{v} - \frac{4}{b} \varphi_4 \eta \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial \mathbf{n}^T}{\partial y} \mathbf{u} - \frac{4}{b} \varphi_2 \eta + \frac{\partial \mathbf{n}^T}{\partial x} \mathbf{v} - \frac{4}{a} \varphi_3 \xi \end{aligned} \quad (14)$$

are separated into

$$\boldsymbol{\epsilon} = \mathbf{B}^T \mathbf{d} + \mathbf{G}^T \boldsymbol{\varphi} \quad (15)$$

corresponding to the standard part and the enhanced part of the element displacement field.

The potential of the external loads,  $\Pi_L$ , is independent of the internal element degrees of freedom,  $\boldsymbol{\varphi}$ . Therefore introduction of (15) into (11) and minimization with respect to  $\mathbf{d}$  and  $\boldsymbol{\varphi}$  yields

$$\frac{\partial \Pi}{\partial \mathbf{d}} = \mathbf{0} \Rightarrow \left( \int \mathbf{B} \mathbf{E} \mathbf{B}^T dV \right) \mathbf{d} + \left( \int \mathbf{B} \mathbf{E} \mathbf{G}^T dV \right) \boldsymbol{\varphi} = \mathbf{f} \quad (16)$$

$$\frac{\partial \Pi}{\partial \boldsymbol{\varphi}} = \mathbf{0} \Rightarrow \left( \int \mathbf{G} \mathbf{E} \mathbf{B}^T dV \right) \mathbf{d} + \left( \int \mathbf{G} \mathbf{E} \mathbf{G}^T dV \right) \boldsymbol{\varphi} = \mathbf{0} \quad (17)$$

with the load vector  $\mathbf{f}$ .

## 4 Equilibrium

On the element level the evaluation of (17) leads to an expression

$$\boldsymbol{\varphi} = \boldsymbol{\varphi}(\mathbf{d}) \quad (18)$$

Then the element strain field can be expressed by

$$\boldsymbol{\epsilon} = \left( \mathbf{B}_0^T + \mathbf{B}_i^T \right) \mathbf{d} + \mathbf{G}^T \boldsymbol{\varphi}(\mathbf{d}) = \left( \mathbf{B}_0^T + \mathbf{B}_*^T \right) \mathbf{d} \quad (19)$$

where the resulting gradient operator is separated into a constant part,  $\mathbf{B}_0$ , and a linear part,  $\mathbf{B}_*$ .

For the rectangle the operators  $\mathbf{B}_0$ ,  $\mathbf{B}_i$ , and  $\mathbf{G}$  are obtained from (7) and (14), and are

$$\mathbf{B}^T = \mathbf{B}_0^T + \mathbf{B}_i^T = \frac{1}{ab} \begin{pmatrix} b \mathbf{g}_\xi^T & \mathbf{0} \\ \mathbf{0} & a \mathbf{g}_\eta^T \\ a \mathbf{g}_\eta^T & b \mathbf{g}_\xi^T \end{pmatrix} + \frac{1}{ab} \begin{pmatrix} b \mathbf{h}^T \eta & \mathbf{0} \\ \mathbf{0} & a \mathbf{h}^T \xi \\ a \mathbf{h}^T \xi & b \mathbf{h}^T \eta \end{pmatrix} \quad (20)$$

$$\mathbf{G}^T = -\frac{4}{ab} \begin{pmatrix} b \xi & 0 & 0 & 0 \\ 0 & 0 & 0 & a \eta \\ 0 & a \eta & b \xi & 0 \end{pmatrix} \quad (21)$$

Upon substitution for  $\mathbf{B}$  and  $\mathbf{G}$  the evaluation of (17) yields

$$\frac{16}{3ab} \begin{pmatrix} E_1 b^2 & & & 0 \\ & G a^2 & & \\ & & G b^2 & \\ 0 & & & E_1 a^2 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{pmatrix} = \frac{4}{3} \begin{pmatrix} E_2 \mathbf{h}^T \mathbf{v} \\ G \mathbf{h}^T \mathbf{v} \\ G \mathbf{h}^T \mathbf{u} \\ E_2 \mathbf{h}^T \mathbf{u} \end{pmatrix} \quad (22)$$

The solution of (22) leads to the coefficients  $\boldsymbol{\varphi}$

$$\boldsymbol{\varphi} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{pmatrix} = \frac{1}{4 E_1 a b} \begin{pmatrix} 0 & E_2 a^2 \mathbf{h}^T \\ 0 & E_1 b^2 \mathbf{h}^T \\ E_1 a^2 \mathbf{h}^T & 0 \\ E_2 b^2 \mathbf{h}^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} \quad (23)$$

Therefore the linear gradient operator  $\mathbf{B}_*$  in (19) is

$$\mathbf{B}_*^T = \frac{1}{E_1 a b} \begin{pmatrix} E_1 b \mathbf{h}^T \eta & -E_2 a \mathbf{h}^T \xi \\ -E_2 b \mathbf{h}^T \eta & E_1 a \mathbf{h}^T \xi \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \quad (24)$$

## 5 Eigensystem

The constant part of the gradient operator yields a rank three stiffness matrix

$$\mathbf{K}_0 = \int_{-1}^1 \int_{-1}^1 \mathbf{B}_0 \mathbf{E} \mathbf{B}_0^T J d\xi d\eta \quad (25)$$

This is the same as the matrix obtained from 1-point integration, since the evaluation of the linear part of the gradient operator at  $\xi = \eta = 0$  yields zero.

Analytical expressions for the non-zero eigenvalues of  $\mathbf{K}_0$ , i.e., the underintegrated four-node rectangular element of unit thickness, are given as follows<sup>23</sup>

$$\begin{aligned} \text{Shear:} \quad \lambda_4 &= \left( \frac{b}{a} + \frac{a}{b} \right) G \\ \text{Extension, Stretching:} \quad \lambda_{5,6} &= \frac{b}{a} E_1 + k_{5,6} \end{aligned} \quad (26)$$

where

$$k_{5,6} = \frac{1}{2} \left( \frac{a}{b} - \frac{b}{a} \right) E_1 \pm \frac{1}{2} \left[ \left( \frac{a}{b} - \frac{b}{a} \right)^2 E_1^2 + 4E_2^2 \right]^{\frac{1}{2}}$$

The eigenvectors corresponding to the above eigenvalues are

$$\begin{aligned} \mathbf{e}_4^T &= \frac{1}{\sqrt{a^2+b^2}} \left( a \mathbf{g}_\eta^T \quad b \mathbf{g}_\xi^T \right) \\ \mathbf{e}_{5,6}^T &= \frac{1}{\sqrt{E_2^2+k_{5,6}^2}} \left( E_2 \mathbf{g}_\xi^T \quad k_{5,6} \mathbf{g}_\eta^T \right) \end{aligned} \quad (27)$$

If the coefficients  $\varphi_1 \dots \varphi_4$  in (19) are set to zero, then the evaluation of the linear operator  $\mathbf{B}_l$  leads to a stabilization matrix which yields the standard stiffness matrix of the rectangle when added to  $\mathbf{K}_0$ . The eigensystem of that rank two stabilization matrix is

$$\begin{aligned} \lambda_7 &= \frac{1}{3} \left( \frac{b}{a} E_1 + \frac{a}{b} G \right) & \mathbf{e}_7^T &= \left( \mathbf{h}^T \quad \mathbf{0} \right) \\ \lambda_8 &= \frac{1}{3} \left( \frac{a}{b} E_1 + \frac{b}{a} G \right) & \mathbf{e}_8^T &= \left( \mathbf{0} \quad \mathbf{h}^T \right) \end{aligned} \quad (28)$$

The eigenvectors that belong to the five non-zero eigenvalues of the rectangle are shown in Figure 2. An eigenvector represents a deformation pattern of the element, while the corresponding eigenvalue refers to the stiffness of that deformation mode.<sup>6</sup>

The standard stiffness matrix ‘locks’, which is reflected by their eigenvalues in (28): as the element aspect ratio,  $a/b$ , increases, not only one but both eigenvalues increase. For the incompressible limit in plane strain, as  $\nu$  approaches 1/2, both eigenvalues become infinite, so that bending performance cannot be approximated by the element.

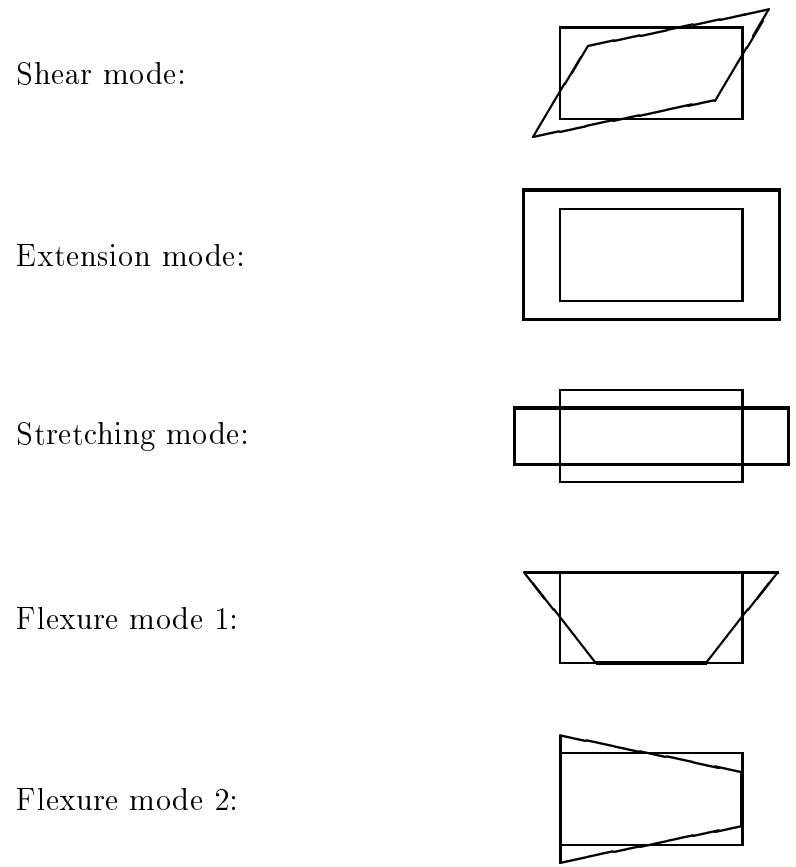


Figure 2: Eigenvectors of the 4-node rectangular element

## 6 Stabilization matrix

In the classical approach, the element stiffness matrix is obtained from the minimization of (11) and the coefficients  $\boldsymbol{\varphi}$  are condensed out on the element level by substituting (17) into (16).<sup>9,21,22</sup> Here, a stabilization matrix is obtained through the evaluation of the linear gradient operator  $\mathbf{B}_*$  in (24).

Being linear in either  $\xi$  or  $\eta$ , the cross terms including the constant and the linear operator are zero

$$\int_{-1}^1 \int_{-1}^1 \mathbf{B}_0 \mathbf{E} \mathbf{B}_*^T J d\xi d\eta = \mathbf{0} \quad \int_{-1}^1 \int_{-1}^1 \mathbf{B}_* \mathbf{E} \mathbf{B}_0^T J d\xi d\eta = \mathbf{0} \quad (29)$$

Therefore the complete stiffness matrix of the rectangle is

$$\mathbf{K} = \mathbf{K}_0 + \mathbf{K}_* = \int_{-1}^1 \int_{-1}^1 \mathbf{B}_0 \mathbf{E} \mathbf{B}_0^T J d\xi d\eta + \int_{-1}^1 \int_{-1}^1 \mathbf{B}_* \mathbf{E} \mathbf{B}_*^T J d\xi d\eta \quad (30)$$

After integration the rank two stabilization matrix becomes

$$\mathbf{K}_* = \frac{1}{3ab} \frac{E_1^2 - E_2^2}{E_1} \begin{pmatrix} b^2 \mathbf{h} \mathbf{h}^T & \mathbf{0} \\ \mathbf{0} & a^2 \mathbf{h} \mathbf{h}^T \end{pmatrix} \quad (31)$$

and the eigensystem of this matrix is

$$\begin{aligned} \lambda_7 &= \frac{1}{3} \frac{E_1^2 - E_2^2}{E_1} \frac{b}{a} & \mathbf{e}_7^T &= \begin{pmatrix} \mathbf{h}^T & \mathbf{0} \end{pmatrix} \\ \lambda_8 &= \frac{1}{3} \frac{E_1^2 - E_2^2}{E_1} \frac{a}{b} & \mathbf{e}_8^T &= \begin{pmatrix} \mathbf{0} & \mathbf{h}^T \end{pmatrix} \end{aligned} \quad (32)$$

For plane stress

$$\begin{aligned} \lambda_7 &= \frac{1}{3} E \frac{b}{a} \\ \lambda_8 &= \frac{1}{3} E \frac{a}{b} \end{aligned} \quad (33)$$

while for plane strain

$$\begin{aligned} \lambda_7 &= \frac{1}{3} \frac{E}{1 - \nu^2} \frac{b}{a} \\ \lambda_8 &= \frac{1}{3} \frac{E}{1 - \nu^2} \frac{a}{b} \end{aligned} \quad (34)$$

Now for an increasing aspect ratio of the element, only one eigenvalue increases while the other one decreases. Furthermore in plane strain both eigenvalues remain small as  $\nu$  approaches 1/2.

In actual fact these eigenvalues represent the exact stiffness of the flexure mode deformation pattern.

## 7 Equivalence with assumed strain method

For the rectangle it will now be shown that the underintegration and stabilization procedure as derived above yields an element identical to that obtained by the assumed strain method.<sup>9,20</sup>

For that method the standard strain field

$$\boldsymbol{\epsilon}_{std} = \mathbf{B}^T \mathbf{d} \quad (35)$$

is enhanced by writing

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}_{std} + \boldsymbol{\epsilon}_{enh} \quad (36)$$

In the present application the enhanced strains are considered as being derived from the incompatible part of the displacement field (12)

$$\boldsymbol{\epsilon}_{enh} = \mathbf{G}^T \boldsymbol{\varphi} = -\frac{4}{ab} \begin{pmatrix} b\xi & 0 & 0 & 0 \\ 0 & 0 & 0 & a\eta \\ 0 & a\eta & b\xi & 0 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{pmatrix} \quad (37)$$

One basic premise of the assumed strain method is that the independently interpolated stresses and enhanced strains are  $L^2$ -orthogonal in the sense that the integral of their inner product vanishes. It follows in particular from this assumption that the stress field, as obtained from the displacement field (12), is also  $L^2$ -orthogonal to *all* assumed strains  $\tilde{\boldsymbol{\epsilon}}_{enh}$ .<sup>20</sup> Since  $\boldsymbol{\sigma} = \mathbf{E} \boldsymbol{\epsilon}$  this implies that

$$\int \tilde{\boldsymbol{\epsilon}}_{enh}^T \mathbf{E} \boldsymbol{\epsilon} dV = \int \tilde{\boldsymbol{\varphi}}^T \mathbf{G} \mathbf{E} (\mathbf{B}^T \mathbf{d} + \mathbf{G}^T \boldsymbol{\varphi}) dV = 0 \quad (38)$$

which must hold for all  $\tilde{\boldsymbol{\varphi}}$ . Therefore the constraint equation on the enhanced strain field is

$$\left( \int \mathbf{G} \mathbf{E} \mathbf{B}^T dV \right) \mathbf{d} + \left( \int \mathbf{G} \mathbf{E} \mathbf{G}^T dV \right) \boldsymbol{\varphi} = \mathbf{0} \quad (39)$$

This gives the same expressions for  $\varphi_1 \dots \varphi_4$  as (17), and therefore the equivalence of the assumed strain approach with the derived stabilization procedure is proved.

## 8 Conclusion

A simple displacement based solution for the stabilization of the rectangular quadrilateral element has been developed. The equivalence with the method of assumed strains was shown. The results have been given in terms of the eigensystem of the element. For the special case of a rectangular element, any proposed rigorous quadrilateral formulation should exhibit the described eigensystem.

The underlying theory serves as a basis to develop a purely displacement based quadrilateral that approaches the accuracy and quality of existing mixed element formulations.<sup>24</sup>

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