AFFINE-APPROXIMATE FINITE ELEMENT METHODS AND STABILIZATION TECHNIQUES IN ELASTICITY

B. Daya Reddy
Faculty of Science
University of Cape Town
7701 Rondebosch
South Africa
bdr@science.uct.ac.za

Abstract The purpose of this work is to report on a method for overcoming the effects of severe mesh distortion in finite element analyses. The method entails the replacement of arbitrary quadrilaterals, in computations, by paralellograms that re closes to the original elements, in a rigorous sense. The resulting method is referred to as an affinie-approximate method, and is developed in the context of enhanced assumed strains with low-order elements. Theoretical results on convergence of the method are presented, and its extension to problems inviolving Mindlin-Reissner plates is discussed, with examples presented.

Keywords: equivalent parallelogram, affine approximations, four-noded quadrilateral, error estimates

1. Introduction

The inherent economy of four-noded quadrilaterals in two dimensions, and eight-noded hexahedral elements in three, results in these being popular choices of elements in finite element analyses. Unfortunately, they are not without their drawbacks. In problems of solid mechanics in which bending deformations dominate, analyses based on these elements exhibit poor accuracy, at least when coarse meshes are used. In addition, in the incompressible limit, or when the compressibility is small, locking behaviour is experienced.

There is a vast literature that is devoted to the construction of methods which are intended to overcome these problems, while retaining the advantages of using low-order elements. Remedies include the use of underintegration and stabilization (see [5] and [8]). Another popular approach is that of enhanced assumed strains [18], which contains as a special case the nonconforming method of incompatible modes [23, 20]. The method has been successfully extended to nonlinear problems (see, for example, [19]). REDDY AND SIMO [15] have shown, for linear problems and for affine elements, that the enhanced strain method is stable and convergent, while ARUNAKIRINATHAR AND REDDY [2] have extended that work to include the case of arbitrary quadrilaterals. In more recent work [6], it has been shown that the method converges uniformly in the incompressible limit.

The method of enhanced assumed strains exhibits a decline in accuracy with increase in element distortions. This problem leads to the notion of replacing the arbitrary quadrilateral by the affine element (a parallelogram) that is closest to it, in a manner that can be made precise. Such an element is known as the equivalent parallelogram in two dimensions, and the equivalent parallelipiped in three. This set of ideas has been proposed, and then tested numerically, first in the context of problems of linear elasticity in [12], and subsequently for problems involving nonlinearly elastic materials, in [16, 17]. In all cases the numerical results are encouraging, and suggest a significant improvement in effiency and accuracy when this approach is used, particularly in circumstances in which element distortions are significant.

The purpose of this work is, first, to report on an analysis that is aimed at verifying the good approximation properties of affine-approximate finite element methods. The key result is that the method converges at the optimal rate provided that the element distortion is of the order of mesh size, in the asymptotic limit. These results are presented in greater detail in [3]. The second aim of this work is to present results on the extension of these methods to the analysis of Mindlin-Reissner plates, for which shear locking is a major challenge in the thin-plate limit. Preliminary results in this regard have appeared in [14], and some further results are presented here.

2. Formulation of the problem

Let Ω be a bounded domain in $\mathbb{R}^d$ $(d = 2, 3)$. We will make use of the space $L^2(\Omega)$ of square-integrable functions defined on Ω. The inner product and norm on this space are denoted respectively by $(\cdot,\cdot)_0$ and $\|\cdot\|_0$. We recall also the definition of the Sobolev space $H^1(\Omega)$, which is the Hilbert space of functions which, together with their first generalised derivatives, are members of $L^2(\Omega)$. We also define the space $H^1_0(\Omega)$ of functions in $H^1(\Omega)$ which vanish on the boundary, in the sense of traces. The seminorm $|\cdot|_1$ is a norm on $H^1_0(\Omega)$, equivalent to the standard norm $\|\cdot\|_1$.

Denote by $V := [H^1_0(\Omega)]^d$ the space of admissible displacements, and define the bilinear form $a(\cdot,\cdot)$ and linear functional $\ell(\cdot)$ by

$$a : V \times V \to \mathbb{R}, \qquad a(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \mathbb{C}\boldsymbol{\epsilon}(\boldsymbol{u}) : \boldsymbol{\epsilon}(\boldsymbol{v})\, d\boldsymbol{x}, \tag{1}$$

$$\ell : V \to \mathbb{R}, \qquad \ell(\boldsymbol{v}) = \int_\Omega \boldsymbol{b}\cdot\boldsymbol{v}\, d\boldsymbol{x}. \tag{2}$$

$\boldsymbol{\epsilon}$ is the infinitesimal strain tensor, $\boldsymbol{u}$ is the displacement vector, $\mathbb{C}$ is the fourth-order elasticity tensor, and $\boldsymbol{b}$ is a prescribed body force vector. For convenience we assume that the displacement satisfies a homogeneous Dirichlet boundary condition.

The properties of $\mathbb{C}$ guarantee that $a(\cdot,\cdot)$ is symmetric, continuous, and V-elliptic. The standard variational problem is as follows.

PROBLEM S. Given $\boldsymbol{b} \in V'$, find $\boldsymbol{u} \in V$ which satisfies

$$a(\boldsymbol{u},\boldsymbol{v}) = \ell(\boldsymbol{v}) \tag{3}$$

for all $\boldsymbol{v} \in V$. It is well known (see, for example, [7]) that Problem S has a unique solution, which depends continuously on the data.

Finite element approximations. We confine attention to plane situations, so that $d = 2$. The domain Ω is assumed to be polygonal, and a finite element mesh $\mathcal{T}$ of quadrilateral elements is constructed on Ω in the usual manner. A typical element K in $\mathcal{T}$ is generated by an isoparametric map F from a reference element $\hat{K} \equiv (-1,1) \times (-1,1)$. The mesh parameter h is defined by $h = \max_{K\in\mathcal{T}} \sup\{|\boldsymbol{x} - \boldsymbol{y}| : \boldsymbol{x}, \boldsymbol{y} \in K\}$. We define basis functions $\hat{N}_A$ $(A = 1, \ldots, 4)$ on $\hat{K}$ by

$$\hat{N}_A(\boldsymbol{\xi}) = \tfrac{1}{4}(1 + \xi\xi_A)(1 + \eta\eta_A),$$

where $\boldsymbol{\xi}_A \equiv (\xi_A, \eta_A)$ are the nodal coordinates on $\hat{K}$, with $(\xi_1 \cdots \xi_4) = (1\ -1\ -1\ 1)$ and $(\eta_1 \cdots \eta_4) = (1\ 1\ -1\ -1)$. Denote by Q_1 the space of bilinear functions spanned by $\hat{N}_A$. Then it is convenient to express the map F in the form

$$F : \hat{K} \to K, \qquad \boldsymbol{x} = F(\boldsymbol{\xi}) = \sum_{A=1}^{4} \boldsymbol{x}_A \hat{N}_A(\boldsymbol{\xi}) \tag{4}$$

in which $\boldsymbol{x}_A$ are the nodal points of K. The Jacobian matrix and determinant are denoted by $\boldsymbol{J}$ and j.

We define $V^h = \{\boldsymbol{v}_h \in V : (v_h)_i|_K \circ F \in Q_1\}$.

The equivalent parallelogram. The equivalent parallelogram associated with a quadrilateral is obtained by perturbing the quadrilateral to obtain the parallelogram that is closest to the quadrilateral, in a precise sense. It has been shown in [1] that the equivalent parallelogram $\tilde{K}$ associated with K is defined by the *affine* map $\tilde{F}$ obtained simply by discarding the bilinear terms in (4). That is, if we define the vector $\boldsymbol{k}$ by $\boldsymbol{k} = \frac{1}{4}(\boldsymbol{x}_1 - \boldsymbol{x}_2 + \boldsymbol{x}_3 - \boldsymbol{x}_4)$, then the map $\tilde{F}$ may be expressed in the form

$$\tilde{F}(\boldsymbol{\xi}) = F(\boldsymbol{\xi}) - \boldsymbol{k}\xi\eta = \sum_{A=1}^{4} N_A(\boldsymbol{\xi})\tilde{\boldsymbol{x}}_A$$

in which the nodal points $\tilde{\boldsymbol{x}}_A$ of the equivalent parallelogram are defined by

$$\tilde{\boldsymbol{x}}_A = \tfrac{3}{4}\boldsymbol{x}_A + \tfrac{1}{4}(\boldsymbol{x}_{A+1} - \boldsymbol{x}_{A+2} + \boldsymbol{x}_{A+3}), \quad A = 1, \dots, 4 \text{ (modulo } 4).$$

These notions are illustrated in Figure 1.

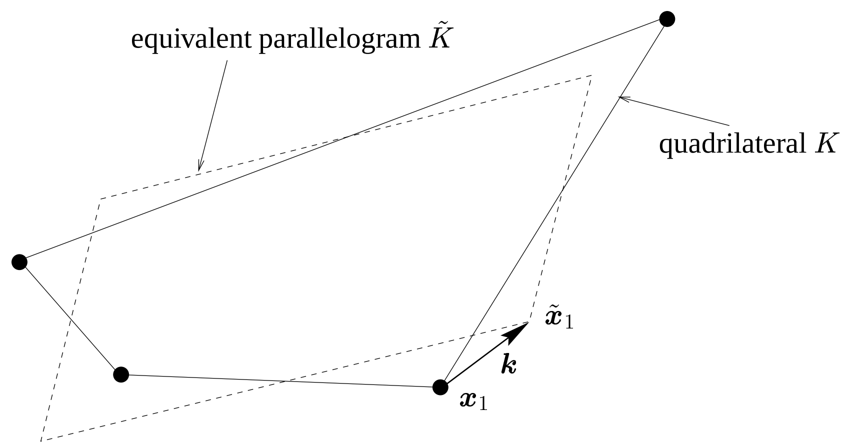

Figure 1. The equivalent parallelogram associated with a quadrilateral

Suppose that the affine map from $\hat{K}$ to $\tilde{K}$ takes the form $\tilde{\boldsymbol{x}} = \boldsymbol{C}\boldsymbol{\xi} + \boldsymbol{c}$, in which $\boldsymbol{C}$ and $\boldsymbol{c}$ are respectively a constant matrix and vector; then the *distortion parameter* τ_K for element K is defined by

$$\tau_K = |\boldsymbol{C}^{-1}\boldsymbol{k}|. \tag{5}$$

The distortion parameter τ associated with a finite element mesh is defined by $\tau = \max_{K\in\mathcal{T}} |\tau_K|$. We define an *$h$-regular mesh* to be a finite element mesh for which $\tau = O(h)$.

The enhanced assumed strain problem. In this formulation, proposed by SIMO AND RIFAI [18], the discrete strain $\boldsymbol{\epsilon}_h$ takes the form

$$\boldsymbol{\epsilon}_h = \boldsymbol{\epsilon}(\boldsymbol{u}_h) + \boldsymbol{\eta}_h, \tag{6}$$

the first term on the righthand side being evaluated as in (**??**), while the second term on the righthand side is the enhanced strain, which is required to have the property $\boldsymbol{\eta}_h \to \boldsymbol{0}$ as $h \to 0$.

In order to formulate the problem in weak form it is necessary to add to the spaces already defined the space Γ^h of enhanced strains, which is defined by

$$\Gamma^h := \{\boldsymbol{\gamma} :\ \gamma_{ij} \in L^2(\Omega),\ \gamma_{ji} = \gamma_{ij},\ \int_K \mathbb{C}\boldsymbol{\gamma}|_K \, dxdy = \boldsymbol{0} \text{ for all } K \in \mathcal{T}\}. \tag{7}$$

In practice Γ^h will comprise functions of the form $\boldsymbol{\gamma} = j^{-1}\hat{\boldsymbol{\gamma}}$ on each element, in which the components of $\hat{\boldsymbol{\gamma}}$ are simple polynomials defined on the reference element $\hat{K}$. A consequence of this definition is that $\int_{\hat{K}} \mathbb{C}\hat{\boldsymbol{\gamma}}\, d\xi d\eta = \mathbf{0}$ [2].

We set $\boldsymbol{\phi}_h = (\boldsymbol{u}_h, \boldsymbol{\eta}_h)$ and $\boldsymbol{\psi}_h = (\boldsymbol{v}_h, \boldsymbol{\gamma}_h)$ for $\boldsymbol{u}_h$, $\boldsymbol{v}_h \in V^h$ and $\boldsymbol{\eta}_h$, $\boldsymbol{\gamma}_h \in \Gamma^h$, and define the product space $\Psi^h := V^h \times \Gamma^h$., The bilinear form $A : \Psi^h \times \Psi^h \longrightarrow \mathbb{R}$ is defined by

$$A(\boldsymbol{\phi}_h, \boldsymbol{\psi}_h) = \int_\Omega \mathbb{C}(\boldsymbol{\epsilon}(\boldsymbol{u}_h) + \boldsymbol{\eta}_h) : (\boldsymbol{\epsilon}(\boldsymbol{v}_h) + \boldsymbol{\gamma}_h)\, dxdy. \tag{8}$$

The weak formulation of the problem then takes the following form [15, 18].

PROBLEM E^h. Find $(\boldsymbol{u}_h, \boldsymbol{\eta}_h) \in V^h \times \Gamma^h$ such that

$$A(\boldsymbol{\phi}_h, \boldsymbol{\psi}_h) = \ell(\boldsymbol{v}_h) \quad \text{for all } \boldsymbol{\psi}_h \in \Psi^h. \tag{9}$$

We have the following result [15, 2, 6].

THEOREM 2. *Let $\mathcal{T}$ be a regular mesh of quadrilaterals on a bounded polygonal domain $\Omega \in \mathbb{R}^2$. Let the space V^h be defined by (**??**), and the space Γ^h by (7). Assume, in addition, that*

(a) $\boldsymbol{\epsilon}(V^h) \cap \Gamma^h = \{\mathbf{0}\}$

(b) there exists a constant c_1 with $0 < c_1 < 1$ such that, for any $\boldsymbol{\gamma}_h \in \Gamma^h$, $\|P\boldsymbol{\gamma}_h\|_\Gamma \leq c_1\|\boldsymbol{\gamma}_h\|$, where P is the L^2-orthogonal projection onto $\boldsymbol{\epsilon}(V^h)$.

Then there exists a unique solution to Problem E^h. Furthermore, if $\boldsymbol{u} \in [H^2(\Omega)]^2$, then there exists a constant $C > 0$, independent of h, such that

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_V + \|\boldsymbol{\eta}_h\|_\Gamma \leq Ch|\boldsymbol{u}|_{H^2}.$$

That is, the method is uniformly convergent in the incompressible limit.

Equivalence with underintegration plus stabilization. For the case in which the finite element mesh comprises elements which are parallelograms, it can be shown [12] that, after the enhanced strain degrees of freedom have been condensed out at element level, the stiffness matrix takes the decoupled form $\boldsymbol{K} = \boldsymbol{K}_\mathrm{o} + \boldsymbol{K}_*$, in which $\boldsymbol{K}_0$ is the matrix corresponding to the constant part of the strain, and $\boldsymbol{K}_*$ is the rank two stabilization matrix for the plane strain problem, given by $\boldsymbol{K}_* = \lambda_7\boldsymbol{e}_7\boldsymbol{e}_7^\mathrm{T} + \lambda_8\boldsymbol{e}_8\boldsymbol{e}_8^\mathrm{T}$, in which $\epsilon_i \ \ (i = 1, 2, 3)$ are expressed in terms of the flexural eigenvalues $\lambda_7,\ \lambda_8$, and α.

Affine finite element approximations. An affine-approximate formulation is constructed by replacing integrals over arbitrary quadrilaterals by integrals over their equivalent parallelograms. Since the map from K to $\tilde{K}$ is affine, the associated Jacobian matrix and determinant are constant, and the integrals can be evaluated exactly.

Numerical results presented by KUESSNER AND REDDY [12] in the case of linear elasticity, and by REESE and coworkers [16, 17] for problems involving nonlinear elasticity and finite deformations, show in addition that, when this concept is applied to enhanced assumed strain formulations, the results represent in many cases an improvement over those obtained by the conventional approach.

Define

$$A_{\tilde{K}}(\tilde{\boldsymbol{\chi}}_h, \tilde{\boldsymbol{\psi}}_h) = \int_{\tilde{K}} \tilde{\mathbb{C}}(\tilde{\boldsymbol{\epsilon}}(\tilde{\boldsymbol{w}}_h) + \tilde{\boldsymbol{\beta}}_h) : (\tilde{\boldsymbol{\epsilon}}(\tilde{\boldsymbol{v}}_h) + \tilde{\boldsymbol{\gamma}}_h)\, d\tilde{x} d\tilde{y} \tag{10}$$

and

$$\ell_{\tilde{K}}(\tilde{\boldsymbol{\psi}}_h) = \int_{\tilde{K}} \boldsymbol{b} \cdot \tilde{\boldsymbol{v}}_h \, d\tilde{x} d\tilde{y}, \tag{11}$$

where $\boldsymbol{\chi}_h = (\boldsymbol{w}_h, \boldsymbol{\beta}_h)$,

$$(\tilde{\nabla}\tilde{\boldsymbol{v}})_{ij} = \frac{\partial \tilde{v}_i}{\partial \tilde{x}_j}, \quad \tilde{\boldsymbol{\epsilon}}(\tilde{\boldsymbol{v}}) = \tfrac{1}{2}(\tilde{\nabla}\tilde{\boldsymbol{v}} + [\tilde{\nabla}\tilde{\boldsymbol{v}}]^T),$$

and superposed tildes on other quantities denote transformation to the domain $\tilde{K}$. Set

$$\tilde{A}(\boldsymbol{\chi}_h, \boldsymbol{\psi}_h) = \sum_{K \in \mathcal{T}} A_{\tilde{K}}(\tilde{\boldsymbol{\chi}}_h, \tilde{\boldsymbol{\psi}}_h) \ \text{ and } \ \tilde{\ell}(\boldsymbol{\psi}_h) = \sum_{K \in \mathcal{T}} \ell_{\tilde{K}}(\tilde{\boldsymbol{\psi}}_h). \tag{12}$$

Then the affine-approximate problem is the following.

Problem $\tilde{\mathrm{E}}^h$. Given $\boldsymbol{b} \in V'$, find $\boldsymbol{\chi}_h := (\boldsymbol{w}_h, \boldsymbol{\beta}_h) \in V^h \times \Psi^h$ which satisfy

$$\tilde{A}(\boldsymbol{\chi}_h, \boldsymbol{\psi}_h) = \tilde{\ell}(\boldsymbol{\psi}_h) \ \text{ for all } \boldsymbol{\psi}_h \in \Psi^h. \tag{13}$$

The following has been proved in [3].

THEOREM 3 *Let $\mathcal{T}$ be an h-regular finite element mesh of quadrilaterals, with the maximum distortion of quadrilaterals being bounded in the sense that $\tau \le ch$ for some constant c, independent of h, as $h \to 0$. Let $\boldsymbol{\phi} = (\boldsymbol{u}, \boldsymbol{0}) \in \Psi$, where $\boldsymbol{u}$ is the solution to Problem S. Then Problem $\tilde{E}^h$ has a unique solution $\boldsymbol{\chi}_h$ which satisfies*

$$\|\boldsymbol{\phi} - \boldsymbol{\chi}_h\|_\Psi \le Ch,$$

the constant C depending on the geometry, on the material tensor $\mathbb{C}$, and on $\boldsymbol{u}$, but not on h. □

3. Application to Mindlin-Reissner plates

In this section it is shown how the theory and methods described earlier may be extended to the case of Mindlin-Reissner plates. Consider a plate which ococcupies the domain $\Omega \times (-t/2, t/2)$. The plate midsurface displacement and rotation are denoted respectively by w and $\boldsymbol{\theta}$, and the curvature $\boldsymbol{\kappa}$ and shear strain γ are then given by $\boldsymbol{\kappa} = (\theta_{1,1} \;\; \theta_{2,2} \;\; \theta_{1,2} + \theta_{2,1})^T$ and $\boldsymbol{\gamma} = -\boldsymbol{\theta} + \nabla w$. The Kirchhoff thin plate limit corresponds to the situation in which $\theta_\alpha = w_{,\alpha}$.

With the bending stiffness defined by $\boldsymbol{D}_b = \begin{pmatrix} E_1 & E_2 & 0 \\ E_2 & E_1 & 0 \\ 0 & 0 & G \end{pmatrix}$, where $E_1 = \frac{E}{1-\nu^2}$, $E_2 = \nu E_1$, and $G = \frac{E}{1+\nu}$, E and ν being Young's modulus and Poisson's ratio, respectively, the total potential energy of the plate is given by

$$\Pi = \tfrac{1}{2} \int_A \left[\tfrac{t^3}{12} \boldsymbol{\kappa}^T \boldsymbol{D}_b \boldsymbol{\kappa} + G t \boldsymbol{\gamma}^T \boldsymbol{\gamma} \right] \, dA - \int_A f w \, dA.$$

The use of piecewise bilinear approximations for w and θ leads to shear locking behaviour when the thickness is very small relative to the lateral dimensions of the plate (see, for example, [9]. Various remedies, for example, selective reduced integration [10], assumed strains or mixed interpolation ([13, 11, 4], and linked interpolations [21, 22], have been proposed.

An attempt has also been made, by Simo and Rifai [18] to overcome the problem of locking through an enhanced assumed strain approach, in which the discrete shear strain is given by $\boldsymbol{\gamma}_h = \boldsymbol{\gamma}(w_h, \theta_{1h}, \theta_{2h}) + \boldsymbol{G}\boldsymbol{\alpha}_h$; the displacements and rotations are approximated by bilinear functions on each element, and the matrix $\boldsymbol{G}$ is given by

$$\boldsymbol{G} = \begin{pmatrix} \xi & 0 & \xi\eta & 0 \\ 0 & \eta & 0 & \xi\eta \end{pmatrix} \tag{14}$$

on the reference element. The strain energy for the discrete problem then becomes

$$\tfrac{1}{2} \int_A \left[\tfrac{t^3}{12} \boldsymbol{\kappa}_h^T \boldsymbol{D}_b \boldsymbol{\kappa}_h + G t (\boldsymbol{\gamma}_h + \boldsymbol{G}\boldsymbol{\alpha}_h)^T (\boldsymbol{\gamma}_h + \boldsymbol{G}\boldsymbol{\alpha}_h) \right] \, dA.$$

For arbitrary quadrilaterals it is necessary to use the transformation $\widetilde{\boldsymbol{G}} = (j/j_0)\boldsymbol{J}_0^{-T}\boldsymbol{G}$ in which $\boldsymbol{J}_0$ is the Jacobian at the centroid and $j_0 = \det \boldsymbol{J}_0$. This leads to a method that is identical to that proposed by Hughes and Tezduyar [11] and Bathe and Dvorkin [4] for rectangles.

For arbitrary quadrilaterals the enhanced assumed strain approach exhibits sensitivity to distortions, and it is natural to enquire whether an affine-approximate approach is able to mitigate the effects of significant mesh distortion. The approach is very similar to that described earlier in the context of problems of plane elasticity, and some examples of the performance of this approach may be found in [14].

Further insight into the behaviour of the new element may be gained by carrying out a spectral analysis of a single element. Consider first a square element with unit side length $L = 1$, thickness varying from t =0.1 to t =0.001, modulus of elasticty $E = 10^4$, and Poisson‘s ratio $\nu = 0.3$.

Table 1 shows the dependence of the eigenvalues λ_i on the thickness t,for the standard element, selective reduced integration, and the enhanced assumed strains. Three of the eigenvalues, corresponding to rigid body modes, go to infinity as $t \to 0$. A locking free element should exhibit only two further eigenvalues going to infinity which correspond to warping modes. For the standard element a total of eight eigenvalues become unbounded, in agreement with the known shear locking response of thsi element. Selective reduced integration minimizes the problem of shear locking and exhibits only two eigenvalues going to infinity, over and above those correspoding to rigid body modes. But the fomulation possesses two additional zero eigenvalues, corresponding to two zero energy modes known as the w-hourglass mode and the in-plane twist mode, which indicate the well known rank deficiency of the reduced integration [9]. The enhanced strain method exhibits four eigenvalues going to infinity, thus leading to a formulation that still exhibits mild locking.

TABLE 1. Dependence on eigenvalues on thickness for a square element

Standard		SRI		EAS	
$h = 0.01$	0.001	0.01	0.001	0.01	0.001
48.08*10^2	48.08*10^4	48.08*10^2	48.08*10^4	48.08*10^2	48.08*10^4
48.08*10^2	48.08*10^4	48.08*10^2	48.08*10^4	48.08*10^2	48.08*10^4
28.85*10^2	28.85*10^4	0.12	0.12	28.85*10^2	28.85*10^4
3.21*10^2	3.21*10^4	0.06	0.06	3.21*10^2	3.21*10^4
3.21*10^2	3.21*10^4	0.06	0.06	0.12	0.12
3.21*10^2	3.21*10^4	0.04	0.04	0.06	0.06
1.07*10^2	1.07*10^4	0.04	0.04	0.06	0.06
1.07*10^2	1.07*10^4	0.00	0.00	0.04	0.04
0.06	0.06	0.00	0.00	0.04	0.04
0.00	0.00	0.00	0.00	0.00	0.00
0.00	0.00	0.00	0.00	0.00	0.00
0.00	0.00	0.00	0.00	0.00	0.00

Table 2 reveals significant differences when the node at (1,1) is translated to (3,3). Results are shown for the case t =0.01, and for the standard element, selective reduced integration, enhanced assumed strains, and enhanced asusmed strains with the affine approximation (EAS-AFF). All formulations exhibit the same number of eigenvalues which become unbounded and which are zero, as in the undistorted case, but for enhanced assumed strains the tendency to mild locking is much smaller.

TABLE 2. Eigenvalues for quadrilateral obtained by moving the node at (1,1) to (3,3)

Standard	SRI	EAS	EAS-AFF
144.99*10^2	144.23*10^2	139.11*10^2	144.95*10^2
66.95*10^2	41.67*10^2	51.69*10^2	43.77*10^2
38.21*10^2	0.28	37.94*10^2	27.97*10^2
9.62*10^2	0.13	8.82*10^2	9.57*10^2
9.23*10^2	0.10	0.26	0.27
9.14*10^2	0.06	0.13	0.12
2.95*10^2	0.02	0.06	0.06
2.83*10^2	0.00	0.06	0.03
0.19	0.00	0.02	0.01
0.00	0.00	0.00	0.00
0.00	0.00	0.00	0.00
0.00	0.00	0.00	0.00

Work in progress.. It is clear, from investigations using the basis (14) for the enhanced strain modes, that the resulting shear strains do not satisfy the constraint of continuous transverse shear along the edges for the case of an arbitrary quadrilateral (see, for example, [22] for a discussion of this criterion). Preliminary studies indicate that the citerion is met if $\boldsymbol{G}$ is augmented by at least two modes, so that it takes the form

$$\boldsymbol{G} = \begin{pmatrix} \xi & 0 & \xi\eta & 0 & 1-3\eta^2 & 0 \\ 0 & \eta & 0 & \xi\eta & 0 & 1-3\xi^2 \end{pmatrix}.$$

The consequences of such a modification are currently being investigated.

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