A POSTERIORI ERROR ESTIMATION AND ADAPTIVE SOLUTION OF ELLIPTIC VARIATIONAL INEQUALITIES OF THE SECOND KIND

Viorel Bostan[1], Weimin Han[2], and B.D. Reddy[3]

Abstract. In this paper, we perform an a posteriori error analysis for adaptive finite element solutions of elliptic variational inequalities of the second kind. A general framework for a posteriori error estimates is established by using duality theory in convex analysis. We then turn to an analysis of some particular a posteriori error estimates of residual type. Efficiency of the error estimators are investigated. Numerous numerical examples are included to illustrate the effectiveness of the a posteriori error estimates in adaptive solutions of the variational inequalities.

1 Introduction

The finite element method today is the dominant numerical method for solving most problems in structural and fluid mechanics. It is widely applied to both linear and nonlinear problems. For practical use of the method, one of the most important problems is the assessment of the reliability of a finite element solution. The reliability of the numerical solution hinges on our ability to estimate errors after the solution is computed; such an error analysis is called a posteriori error analysis. A posteriori error estimates provide quantitative information on the accuracy of the solution and are the basis for the development of automatic, adaptive procedures for engineering applications of the finite element method.

The interest in a posteriori error estimation for the finite element method began in the late 1970's. The pioneering work on the topic was done in [4, 5]. Since then, a posteriori error analysis and adaptive computation in the finite element method have attracted many researchers, and a variety of different a posteriori error estimates have been proposed and analyzed. Some comprehensive summary accounts can be found in [1], [6] and [42].

Most of the work so far on a posteriori error analysis has been devoted to ordinary boundary value problems of partial differential equations. In applications, an important family of nonlinear boundary value and initial-boundary value problems is that associated with variational inequalities, that is, problems involving either differential inequalities or inequality boundary conditions. Mechanics is a rich source of variational inequalities (cf. e.g. [37]), and some examples of problems that give rise to variational inequalities are obstacle and contact problems, plasticity and visco-plasticity problems, Stefan problems, unilateral problems of plates and shells, and non-Newtonian flows involving Bingham fluids. An early comprehensive reference on the topic is [14], where many nonlinear boundary value problems in mechanics and physics are formulated and studied in the framework of variational

[1]Department of Mathematics, University of Iowa, Iowa City, IA 52242, U.S.A. E-mail: vbostan@math.uiowa.edu

[2]Department of Mathematics, University of Iowa, Iowa City, IA 52242, U.S.A. E-mail: whan@math.uiowa.edu

[3]Department of Mathematics and Applied Mathematics, University of Cape Town, 7700 Rondebosch, South Africa. E-mail: bdr@maths.uct.ac.za

inequalities. A concise introduction to the mathematical theory of some variational inequalities can be found in [34]. Numerical approximations of general variational inequalities are studied in detail in [17, 18]. Numerical methods for some variational inequalities arising in mechanics are the subject of [29, 30]. Mathematical analysis and numerical approximations of variational inequalities arising in contact mechanics are presented in [33] (for elastic materials) and [28] (for viscoelastic and viscoplastic materials). In [25, 26], elastoplasticity problems are formulated and analyzed in the form of variational inequalities.

Although several standard techniques have been developed to derive and analyze a posteriori error estimates for finite element solutions to problems in the form of variational equations, they do not work directly for a posteriori error analysis of numerical solutions to variational inequalities. Nevertheless, numerous papers can be found on a posteriori error estimation of finite element solutions of obstacle problems, e.g., [2], [11], [31], [35], [36], [43] (these papers consider numerical solutions on convex subsets of finite element spaces), as well as [16], [32] (these papers use a penalty approach for discrete solutions). Obstacle problems are so-called variational inequalities of the first kind, that is, they are inequalities involving smooth functionals and are posed over convex subsets. We also note that a posteriori error estimation is discussed in [8, 9, 40]. However, the arguments in these papers do not lead to the stated error bounds.

In the context of elastoplasticity with hardening, a computable a posteriori error estimate is derived in [10] and [3] for the primal problem, which is a variational inequality of the second kind; that is, the inequality arises as a result of the presence of a non-differentiable functional. These works deal extensively also with a priori estimates, and in the latter work a number of numerical examples are presented.

In this paper, we derive and study some a posteriori error estimates for finite element solutions of elliptic variational inequalities of the second kind. The basic mathematical tool we will use is the duality theory in convex analysis (cf. [15], [44]). The duality theory has been applied to derive efficient a posteriori error estimates for mathematical idealizations of physical and engineering problems (see, e.g., [20], [21]), as well as for some numerical procedures for solving nonlinear problems, such as the regularization techniques in [19], [24] and [27], and the Kačanov iteration method in [22, 23]. In [39, 38], the technique of the duality theory was used to derive a posteriori error estimates of the finite element method in solving boundary value problems of some nonlinear equations. In these papers, the error bounds are shown to converge to zero in the limit; however, no efficiency analysis of the estimates is given.

This paper is organized as follows. In Section 2 we introduce a model elliptic variational inequality of the second kind, on which all our later analysis will be based. In Section 3 we formulate a dual problem for the model. In Section 4 a posteriori error estimates for approximations of the solution of the model problem are derived, based on the use of duality theory. Section 5 contains the finite element discretization of the model problem and a posteriori error estimates for it. In Section 6, we study the efficiency of the error estimators. Finally, in Section 7 we present some numerical examples to illustrate the effectivity of the estimates in adaptive solution of variational inequalities.

We now list some notations used in the paper. Let Ω be a bounded domain in $\mathbb{R}^d$, $d \geq 1$, with Lipschitz boundary $\Gamma = \partial\Omega$. Denote by D an open subset of Ω or Γ. For any open subset ω of Ω

with Lipschitz boundary $\partial\omega$ we denote by $H^m(\omega)$, $L^2(\omega)$ and $L^2(\partial\omega)$ the usual Sobolev and Lebesgue spaces with the standard norms $\|\cdot\|_{m;\omega} := \|\cdot\|_{H^m(\omega)}$, $\|\cdot\|_{0;\omega} := \|\cdot\|_{L^2(\omega)}$ and $\|\cdot\|_{0;\partial\omega} := \|\cdot\|_{L^2(\partial\omega)}$. Also, we will make use of the standard seminorm $|\cdot|_{m,\omega} := \|D^m\cdot\|_{0;\omega}$ on $H^m(\omega)$. We set $H_0^1(\Omega) \equiv \{v \in H^1(\Omega) : v|_\Gamma = 0\}$. Throughout this paper we will use the same notation v to denote both $v \in H^1(\Omega)$ and its trace $\gamma v \in L^2(\Gamma)$ on the boundary. We reserve the symbol γ to denote sides of the finite elements.

2 Model problem

We introduce a model elliptic variational inequality of the second kind in this section. We comment that the ideas and techniques presented in this paper for a posteriori error analysis in solving the model problem can be extended to other elliptic variational inequalities of the second kind. Over the space $V = H^1(\Omega)$, we define

$$a(u,v) = \int_\Omega (\nabla u \cdot \nabla v + u\,v),$$
$$l(v) = \int_\Omega f v,$$
$$j(v) = \int_D g\,|v|,$$

where $f \in L^2(\Omega)$ and $g > 0$ are given. The model problem is the following elliptic variational inequality of the second kind: Find $u \in V$ such that

$$a(u, v-u) + j(v) - j(u) \geq l(v-u) \quad \forall\, v \in V. \tag{2.1}$$

In the case $D \subset \Gamma$, the model we use is a so-called simplified friction problem (cf. [17]) as it can be viewed as a simplified version of a frictional contact problem in linearized elasticity.

Since the bilinear form $a(\cdot,\cdot)$ is symmetric and positive definite, the variational inequality (2.1) is equivalent to the minimization problem: Find $u \in V$ such that

$$J(u) = \inf_{v\in V} J(v), \tag{2.2}$$

where J is the energy functional

$$J(v) = \frac{1}{2}a(v,v) + j(v) - l(v). \tag{2.3}$$

Existence and uniqueness of a solution for the problems (2.1) and (2.2) follow from a classical result (see e.g. [17] and [18]).

In the analysis of a posteriori error estimators later, we will need the following characterization of the solution u of (2.1):

There exists a unique $\lambda \in L^\infty(D)$ such that

$$a(u,v) + \int_D g\,\lambda\,v = l(v) \quad \forall\, v \in V, \tag{2.4}$$

$$|\lambda| \leq 1, \quad \lambda\,u = |u| \quad \text{a.e. in } D. \tag{2.5}$$

The function λ can be viewed as a Lagrange multiplier.

A proof of this characterization in the case $D = \Gamma$ can be found in [17]. The argument there can be extended to the more general situation considered in this paper without a problem; see also the proof of Theorem 5.1.

Denote by χ_D the characteristic function of the set D. It follows from the above characterization that in the case $D \subset \Gamma$, the solution u of (2.1) is the weak solution of the boundary value problem

$$\begin{aligned} -\Delta u + u &= f \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n} + g\lambda\chi_D &= 0 \quad \text{on } \Gamma. \end{aligned}$$

In the case $D \subset \Omega$, the boundary value problem is

$$\begin{aligned} -\Delta u + u + g\lambda\chi_D &= f \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n} &= 0 \quad \text{on } \Gamma. \end{aligned}$$

3 Dual formulation

We now present a dual formulation for the model problem following the framework developed in [15]. The dual formulation will be used in the derivation of a posteriori error estimators for approximate solutions.

Let $Q = (L^2(\Omega))^d \times L^2(\Omega) \times L^2(D)$. Define a function $F : V \times Q \to \overline{\mathbb{R}} \equiv \mathbb{R} \cup \{\pm\infty\}$ by the formula

$$F(v, \boldsymbol{q}) = \int_\Omega \left[\frac{1}{2}(|\boldsymbol{q}_1|^2 + |q_2|^2) - fv\right] + \int_D g\,|q_3|,$$

where $\boldsymbol{q} = (\boldsymbol{q}_1, q_2, q_3) \in Q$. Introduce a linear bounded operator $\Lambda : V \to Q$ by the relation

$$\Lambda v = (\nabla v, v, v|_D) \quad \forall\, v \in V.$$

Then obviously,

$$J(v) = F(v, \Lambda v) \quad \forall\, v \in V.$$

Therefore, the minimization problem (2.2) can be rewritten as: Find $u \in V$ such that

$$F(u, \Lambda u) = \inf_{v \in V} F(v, \Lambda v). \tag{3.1}$$

Let V^* and $Q^* = (L^2(\Omega))^d \times L^2(\Omega) \times L^2(D)$ be the duals of V and Q placed in duality by the pairings $\langle \cdot, \cdot \rangle_V$ and $\langle \cdot, \cdot \rangle_Q$, respectively. The conjugate function F^* of F is defined by

$$F^*(\Lambda^* \boldsymbol{q}^*, -\boldsymbol{q}^*) \equiv \sup_{v \in V, \boldsymbol{q} \in Q} \left\{ \langle \Lambda^* \boldsymbol{q}^*, v \rangle_V - \langle \boldsymbol{q}^*, \boldsymbol{q} \rangle_Q - F(v, \boldsymbol{q}) \right\},$$

where $\Lambda^* : Q^* \to V^*$ is the adjoint of Λ. We have

$$\begin{aligned} F^*(\Lambda^*\boldsymbol{q}^*, -\boldsymbol{q}^*) = \sup_{v\in V, \boldsymbol{q}\in Q} \Bigg\{ & \int_\Omega \left[\boldsymbol{q}_1^* \cdot \nabla v + (q_2^* + f)\, v\right] + \int_D q_3^* v \\ & - \int_\Omega \left(\frac{1}{2}|\boldsymbol{q}_1|^2 + \boldsymbol{q}_1^* \cdot \boldsymbol{q}_1\right) - \int_\Omega \left(\frac{1}{2}|q_2|^2 + q_2^* q_2\right) \\ & - \int_D (q_3^* q_3 + g\, |q_3|) \Bigg\}. \end{aligned} \tag{3.2}$$

It can easily be verified that

$$\sup_{\boldsymbol{q}_1 \in (L^2(\Omega))^d} \left\{ -\int_\Omega \left(\frac{1}{2}|\boldsymbol{q}_1|^2 + \boldsymbol{q}_1^* \cdot \boldsymbol{q}_1\right)\right\} = \int_\Omega \frac{1}{2}|\boldsymbol{q}_1^*|^2, \tag{3.3}$$

$$\sup_{q_2 \in L^2(\Omega)} \left\{ -\int_\Omega \left(\frac{1}{2}|q_2|^2 + q_2^* q_2\right)\right\} = \int_\Omega \frac{1}{2}|q_2^*|^2, \tag{3.4}$$

$$\sup_{q_3 \in L^2(D)} \left\{ -\int_D (q_3^* q_3 + g\, |q_3|)\right\} = \begin{cases} 0, & \text{if } |q_3^*| \le g \text{ a.e. in } D, \\ +\infty, & \text{otherwise.} \end{cases} \tag{3.5}$$

Define the set of admissible functions

$$Q_{f,g}^* = \left\{ \boldsymbol{q}^* \in Q^* : \int_\Omega \left[\boldsymbol{q}_1^* \cdot \nabla v + (q_2^* + f)\, v\right] + \int_D q_3^* v = 0 \;\forall\, v \in V,\ |q_3^*| \le g \text{ a.e. in } D \right\}. \tag{3.6}$$

Then by using (3.3)–(3.6) we finally get

$$F^*(\Lambda^*\boldsymbol{q}^*, -\boldsymbol{q}^*) = \begin{cases} \int_\Omega \frac{1}{2}\left(|\boldsymbol{q}_1^*|^2 + |q_2^*|^2\right), & \text{if } \boldsymbol{q}^* \in Q_{f,g}^*, \\ +\infty, & \text{otherwise.} \end{cases} \tag{3.7}$$

Formally, $\boldsymbol{q}^* = (\boldsymbol{q}_1^*, q_2^*, q_3^*)$ belongs to the admissible set $Q_{f,g}^*$ if and only if

$$\begin{aligned} \operatorname{div} \boldsymbol{q}_1^* - q_2^* - q_3^* \chi_D &= f \quad \text{in } \Omega, \\ \boldsymbol{q}_1^* \cdot \boldsymbol{n} &= 0 \quad \text{on } \Gamma, \\ |q_3^*| &\le g \quad \text{in } D \end{aligned}$$

when $D \subset \Omega$, or

$$\begin{aligned} \operatorname{div} \boldsymbol{q}_1^* - q_2^* &= f \quad \text{in } \Omega, \\ \boldsymbol{q}_1^* \cdot \boldsymbol{n} + q_3^* \chi_D &= 0 \quad \text{on } \Gamma, \\ |q_3^*| &\le g \quad \text{on } D \end{aligned}$$

when $D \subset \Gamma$.

The dual problem of (3.1) can now be stated: Find $\boldsymbol{p}^* \in Q_{f,g}^*$ such that

$$-F^*(\Lambda^*\boldsymbol{p}^*, -\boldsymbol{p}^*) = \sup_{\boldsymbol{q}^* \in Q_{f,g}^*} \left\{ -F^*(\Lambda^*\boldsymbol{q}^*, -\boldsymbol{q}^*)\right\}. \tag{3.8}$$

Existence of solutions of the problems (3.1) and (3.8) is assured by the following theorem (cf. [15]).

Theorem 3.1 *Assume*

(1) V *is a reflexive Banach space*, Q *a normed space*;

(2) $F : V \times Q \to \overline{\mathbb{R}} \equiv \mathbb{R} \cup \{\pm\infty\}$ *is a proper, lower semi-continuous, convex function*;

(3) $\Lambda : V \to Q$ *is a linear bounded operator together with its adjoint* $\Lambda^* : Q^* \to V^*$;

(4) $\exists\, u_0 \in V$ *such that* $F(u_0, \Lambda u_0) < \infty$ *and* $\boldsymbol{q} \mapsto F(u_0, \boldsymbol{q})$ *is continuous at* Λu_0;

(5) $F(v, \Lambda v) \to +\infty$ *as* $\|v\| \to \infty \quad \forall\, v \in V$.

Then the problem (3.1) *has a solution* $u \in V$*, its dual* (3.8) *has a solution* $\boldsymbol{p}^* \in Q^*$*, and*

$$F(u, \Lambda u) = -F^*(\Lambda^* \boldsymbol{p}^*, -\boldsymbol{p}^*). \tag{3.9}$$

Both the model problem (3.1) and its dual (3.8) have unique solutions, due to the strict convexity of the functions $v \mapsto F(v, \Lambda v)$ over V and $\boldsymbol{q} \mapsto F^*(\Lambda^* \boldsymbol{q}^*, -\boldsymbol{q}^*)$ over $Q^*_{f,g}$.

4 A posteriori error estimates

Let $u \in V$ be the unique solution of (2.1) and $w \in V$ an approximation of u. In this section, we derive some a posteriori estimates for the error $u - w$. The error bounds are computable from the (known) approximant w. Later in the paper, w will be taken as the finite element solution of the variational inequality.

By using (2.1) and (2.3) we obtain

$$\begin{aligned}
\frac{1}{2} a(u - w, u - w) &= \frac{1}{2} a(w, w) - a(u, w) + \frac{1}{2} a(u, u) \\
&= \frac{1}{2} a(w, w) - a(u, w - u) - \frac{1}{2} a(u, u) \\
&\le \frac{1}{2} a(w, w) + j(w) - j(u) - l(w - u) - \frac{1}{2} a(u, u) \\
&= J(w) - J(u).
\end{aligned}$$

On the other hand, let $\boldsymbol{p}^*$ be the solution of the dual problem (3.8). Relation (3.9) implies

$$J(u) = F(u, \Lambda u) = -F^*(\Lambda^* \boldsymbol{p}^*, -\boldsymbol{p}^*) \ge -F^*(\Lambda^* \boldsymbol{q}^*, -\boldsymbol{q}^*) \quad \forall\, \boldsymbol{q}^* \in Q^*_{f,g}.$$

Therefore,

$$\begin{aligned}
\frac{1}{2} a(u - w, u - w) &\le J(w) + F^*(\Lambda^* \boldsymbol{q}^*, -\boldsymbol{q}^*) \\
&\qquad + \int_\Omega \frac{1}{2}(|\boldsymbol{r}_1^*|^2 + |r_2^*|^2) - \int_\Omega \frac{1}{2}(|\boldsymbol{r}_1^*|^2 + |r_2^*|^2) \\
&= \int_\Omega \left[\frac{1}{2}(|\nabla w + \boldsymbol{r}_1^*|^2 + |w + r_2^*|^2) - \nabla w \cdot \boldsymbol{r}_1^* - w r_2^* - f w \right] \\
&\qquad + \int_D g\, |w| + \int_\Omega \frac{1}{2}(|\boldsymbol{q}_1^*|^2 - |\boldsymbol{r}_1^*|^2 + |q_2^*|^2 - |r_2^*|^2) \\
&\qquad \forall\, \boldsymbol{q}^* = (\boldsymbol{q}_1^*, q_2^*, q_3^*) \in Q^*_{f,g},\ \forall\, \boldsymbol{r}^* = (\boldsymbol{r}_1^*, r_2^*, r_3^*) \in Q^*.
\end{aligned} \tag{4.1}$$

It follows immediately from (3.6) that for any $\boldsymbol{q}^* = (\boldsymbol{q}_1^*, q_2^*, q_3^*) \in Q^*_{f,g}$,

$$\int_\Omega (\boldsymbol{q}_1^* \cdot \nabla w + q_2^* w) + \int_D q_3^* \, w = -\int_\Omega f \, w \quad \forall \, w \in V. \tag{4.2}$$

Using (4.2) and regrouping terms in (4.1) we find that

$$\begin{aligned}
&\frac{1}{2} a(u-w, u-w) \\
&\quad \le \int_\Omega \frac{1}{2}(|\nabla w + \boldsymbol{r}_1^*|^2 + |w + r_2^*|^2) + \int_\Omega \frac{1}{2}(|\boldsymbol{q}_1^* - \boldsymbol{r}_1^*|^2 + |q_2^* - r_2^*|^2) \\
&\quad\quad + \int_\Omega [(\boldsymbol{q}_1^* - \boldsymbol{r}_1^*) \cdot (\nabla w + \boldsymbol{r}_1^*) + (q_2^* - r_2^*)(w + r_2^*)] + \int_D (g\,|w| + q_3^* w) \\
&\quad \le \int_\Omega (|\nabla w + \boldsymbol{r}_1^*|^2 + |w + r_2^*|^2) + \int_\Omega (|\boldsymbol{q}_1^* - \boldsymbol{r}_1^*|^2 + |q_2^* - r_2^*|^2) \\
&\quad\quad + \int_D (g\,|w| + q_3^* w) \quad \forall \, \boldsymbol{q}^* \in Q^*_{f,g}, \quad \forall \, \boldsymbol{r}^* \in Q^*.
\end{aligned}$$

Thus, we have established the following result.

Theorem 4.1 *Let $u \in V$ be the unique solution of (2.1), and $w \in V$ an approximation of u. Then the following estimate holds for any $\boldsymbol{r}^* \in Q^*$:*

$$\begin{aligned}
\frac{1}{2} a(u-w, u-w) \le &\int_\Omega (|\nabla w + \boldsymbol{r}_1^*|^2 + |w + r_2^*|^2) \\
&+ \inf_{\boldsymbol{q}^* \in Q^*_{f,g}} \left\{ \int_\Omega (|\boldsymbol{q}_1^* - \boldsymbol{r}_1^*|^2 + |q_2^* - r_2^*|^2) + \int_D (g\,|w| + q_3^* w) \right\}.
\end{aligned} \tag{4.3}$$

Let us now deal with the second term II on the right side of the estimate (4.3). First, from the definition (3.6) it follows that

$$\begin{aligned}
II &\equiv \inf_{\boldsymbol{q}^* \in Q^*_{f,g}} \left\{ \int_\Omega (|\boldsymbol{q}_1^* - \boldsymbol{r}_1^*|^2 + |q_2^* - r_2^*|^2) + \int_D (g\,|w| + q_3^* w) \right\} \\
&= \inf_{\boldsymbol{q}_1^*, q_2^*, |q_3^*| \le g} \sup_{v \in V} \left\{ \int_\Omega (|\boldsymbol{q}_1^* - \boldsymbol{r}_1^*|^2 + |q_2^* - r_2^*|^2) + \int_D (g\,|w| + q_3^* w) \right. \\
&\qquad \left. + \int_\Omega [\boldsymbol{q}_1^* \cdot \nabla v + (q_2^* + f)\, v] + \int_D q_3^* v \right\}.
\end{aligned}$$

Substitute $\boldsymbol{q}_1^* - \boldsymbol{r}_1^*$ by $\boldsymbol{q}_1^*$, $q_2^* - r_2^*$ by q_2^* and regroup the terms to get

$$\begin{aligned}
II &= \inf_{\boldsymbol{q}_1^*, q_2^*, |q_3^*| \le g} \sup_{v \in V} \left\{ \int_\Omega (|\boldsymbol{q}_1^*|^2 + \boldsymbol{q}_1^* \cdot \nabla v) + \int_\Omega (|q_2^*|^2 + q_2^* v) \right. \\
&\qquad \left. + \int_\Omega [\boldsymbol{r}_1^* \cdot \nabla v + (r_2^* + f)\, v] + \int_D [g\,|w| + q_3^*(w + v)] \right\} \\
&= \inf_{|q_3^*| \le g} \sup_{v \in V} \left\{ \int_\Omega \left[-\frac{1}{4}(|\nabla v|^2 + v^2) + \boldsymbol{r}_1^* \cdot \nabla v + (r_2^* + f) v \right] + \int_D q_3^* v \right. \\
&\qquad \left. + \int_D (g\,|w| + q_3^* w) \right\}.
\end{aligned}$$

Define the residual

$$\mathcal{R}(q_3^*, \boldsymbol{r}^*) = \sup_{v \in V} \frac{1}{\|v\|_V} \left\{ \int_\Omega \left[\boldsymbol{r}_1^* \cdot \nabla v + (r_2^* + f)\, v \right] + \int_D q_3^* v \right\}. \tag{4.4}$$

Then,

$$\begin{aligned} II &\le \inf_{|q_3^*| \le g} \sup_{v \in V} \left\{ -\frac{1}{4} \|v\|_V^2 + \mathcal{R}(q_3^*, \boldsymbol{r}^*)\, \|v\|_V + \int_D (g\,|w| + q_3^* w) \right\} \\ &= \inf_{|q_3^*| \le g} \left\{ \mathcal{R}(q_3^*, \boldsymbol{r}^*)^2 + \int_D (g\,|w| + q_3^* w) \right\}. \end{aligned}$$

This last estimate is combined with Theorem 4.1 to establish the next result.

Theorem 4.2 *Let $u \in V$ be the unique solution of (2.1), and $w \in V$ an approximation of u. Then the following error bound*

$$\begin{aligned} \frac{1}{2} a(u - w, u - w) \le \int_\Omega &(|\nabla w + \boldsymbol{r}_1^*|^2 + |w + r_2^*|^2) \\ &+ \inf_{|q_3^*| \le g} \left\{ \mathcal{R}(q_3^*, \boldsymbol{r}^*)^2 + \int_D (g\,|w| + q_3^* w) \right\} \\ &\forall\, \boldsymbol{r}^* = (\boldsymbol{r}_1^*, r_2^*, r_3^*) \in Q^* \end{aligned} \tag{4.5}$$

is valid, where the residual $\mathcal{R}(q_3^*, \boldsymbol{r}^*)$ is defined by (4.4).

With the special selection $\boldsymbol{r}_1^* = -\nabla w$, $r_2^* = -w$, estimate (4.5) leads to the following corollary.

Corollary 4.3 *Let $u \in V$ be the unique solution of (2.1) and $w \in V$ an approximation. Then the following a posteriori error estimate holds:*

$$\begin{aligned} \sqrt{\frac{1}{2} a(u - w, u - w)} \le \mathcal{R} \equiv \inf_{|q_3^*| \le g} \Bigg\{ &\left[\sup_{v \in V} \frac{1}{\|v\|_V} \left(\int_\Omega [-\nabla w \cdot \nabla v + (-w + f) v] + \int_D q_3^* v \right) \right]^2 \\ &+ \int_D (g\,|w| + q_3^* w) \Bigg\}^{1/2}. \end{aligned} \tag{4.6}$$

Although it is possible to derive (4.6) through other approaches, we comment that Theorem 4.2 provides a general framework for various a posteriori error estimates with different choices of the auxiliary variable $\boldsymbol{r}^*$. Some of the choices will be considered in forthcoming papers.

In the limiting case $g = 0$, Problem (2.1) reduces to the variational equation

$$u \in V, \quad a(u, v) = l(v) \quad \forall\, v \in V.$$

We observe that correspondingly, the estimate (4.6) reduces to the familiar form

$$\sqrt{\frac{1}{2} a(u - w, u - w)} \le \sup_{v \in V} \frac{1}{\|v\|_V} \int_\Omega \left[(f - w) v - \nabla w \cdot \nabla v \right],$$

which is a starting point in deriving some a posteriori error estimators for Galerkin approximations of linear elliptic partial differential equations (cf. [1]).

5 Finite element approximation and a posteriori error bounds

For simplicity, we suppose that Ω has a polyhedral boundary Γ. In order to define the finite element method for (2.1) we introduce a family of finite element spaces $V_h \subset V$, which consist of continuous piecewise polynomial functions defined on partitions $\mathcal{P}_h$ of Ω. For every element $K \in \mathcal{P}_h$, let h_K be the diameter of K and ρ_K be the diameter of the largest ball inscribed into K. For a side (or surface; in the following, we use the word "side" to refer to both possibilities) γ of the element K, we denote by h_γ the diameter of γ. We shall assume that the family of partitions $\mathcal{P}_h$, $h > 0$, satisfies the shape regularity assumption, i.e. the ratio h_K/ρ_K is uniformly bounded over the whole family by a constant C. Note that the shape regularity assumption does not require that the elements be of comparable size and thus locally refined meshes are allowed. We also assume that the finite element partitions are compatible with the set D, i.e., in the case $D \subset \Omega$, $\overline{D}$ is completely covered by a group of elements, while in the case $D \subset \Gamma$, if $\partial K \cap D \neq \emptyset$ for some element K, then $\partial K \cap \overline{D}$ consists of complete sides or surfaces of ∂K. We will use $\partial\mathcal{P}_h$ for the set of the element sides, $\partial\mathcal{P}_{h,\Gamma}$ for the subset of the element sides lying on the boundary Γ, and $\partial\mathcal{P}_{h,0} = \partial\mathcal{P}_h \backslash \partial\mathcal{P}_{h,\Gamma}$.

The patch $\widetilde{K}$ associated with any element K from a partition $\mathcal{P}_h$ consists of all elements sharing at least one vertex with K i.e. $\widetilde{K} = \mathrm{int}\big(\bigcup\{K' \in \mathcal{P}_h : \overline{K'} \cap \overline{K} \neq \emptyset\}\big)$. Similarly, for any side $\gamma \in \partial\mathcal{P}_h$, the patch $\widetilde{\gamma}$ consists of the elements sharing γ as a common side. It should be noted that in the case when the side γ lies on the boundary Γ, the patch $\widetilde{\gamma}$ consists of only one element. Let $\Pi_h : V \to V_h$ be the locally regularized interpolation operator introduced by Bernardi and Girault in [7], which is in fact a version of the interpolation operator of Clément [13]. It is known that the local bounds

$$\|v - \Pi_h v\|_{0;K} \le C\, h_K \|v\|_{1;\widetilde{K}}, \tag{5.1}$$

$$\|v - \Pi_h v\|_{0;\partial K} \le C\, h_K^{1/2} \|v\|_{1;\widetilde{K}}, \tag{5.2}$$

hold for all $v \in V$ and all $K \in \mathcal{P}_h$, where C is a positive constant independent of v and h_K.

The discrete formulation corresponding to the variational inequality (2.1) defined by the Galerkin finite element method reads: Find $u_h \in V_h$ such that

$$a(u_h, v_h - u_h) + j(v_h) - j(u_h) \ge l(v_h - u_h) \quad \forall\, v_h \in V_h. \tag{5.3}$$

We need the following characterization of the finite element solution, similar to that of the solution of the continuous problem.

Theorem 5.1 *The unique solution $u_h \in V_h$ of the discretized problem (5.3) is characterized by the existence of $\lambda_h \in L^\infty(D)$ such that*

$$a(u_h, v_h) + \int_D g\, \lambda_h v_h = l(v_h) \quad \forall\, v_h \in V_h, \tag{5.4}$$

$$|\lambda_h| \le 1, \quad \lambda_h u_h = |u_h| \quad \text{a.e. in } D. \tag{5.5}$$

Proof. Let us give a proof of the result for the case $D = \Gamma$. The other cases of D can be treated similarly. First, we will prove that relation (5.3) implies (5.4) and (5.5). Taking first $v_h = 0$ and then $v_h = 2u_h$ in (5.3), we obtain

$$a(u_h, u_h) + \int_\Gamma g\, |u_h| = l(u_h), \tag{5.6}$$

Together with (5.6) the relation (5.3) leads to

$$|l(v_h) - a(u_h, v_h)| \le \int_\Gamma g\,|v_h| \quad \forall\, v_h \in V_h. \tag{5.7}$$

Write $V_h = V_h^0 \oplus V_h^\perp$, where $V_h^0 = V_h \cap H_0^1(\Omega)$ and $V_h^\perp$ is the orthogonal complement of V_h in $H_0^1(\Omega)$. It follows from (5.7) that $l(v_h) - a(u_h, v_h) = 0\ \forall\, v_h \in V_h^0$. Notice that the trace operator from $V_h^\perp$ onto $V_h^\perp|_\Gamma \subset L^1(\Gamma)$ is an isomorphism. Therefore, the mapping $L(v_h) \equiv l(\widetilde{v}_h) - a(u_h, \widetilde{v}_h)$ can be viewed as a linear functional on $V_h^\perp|_\Gamma$, where $\widetilde{v}_h$ is any element from the space V_h whose trace on Γ is v_h. It follows from (5.7) that

$$|L(v_h)| \le \int_\Gamma g\,|v_h| \quad \forall\, v_h \in V_h^\perp|_\Gamma. \tag{5.8}$$

Thus, by the Hahn-Banach Theorem the functional $L(v_h)$ can be extended to $L(v)$ on $L^1(\Gamma)$ and so there exists $\lambda_h \in L^\infty(\Gamma)$ such that

$$L(v) = \int_\Gamma \lambda_h g\, v \quad \forall\, v \in L^1(\Gamma)$$

and $|\lambda_h| \le 1$ a.e. on Γ, from which (5.4) follows. Taking now $v_h = u_h$ in relation (5.4), we have

$$a(u_h, u_h) + \int_\Gamma g\,\lambda_h u_h = l(u_h),$$

and using (5.6) we get

$$\int_\Gamma g\,(|u_h| - \lambda_h u_h) = 0.$$

Since $|\lambda_h| \le 1$ a.e. on Γ, we must have $|u_h| = \lambda_h u_h$ a.e. on Γ. This completes the proof of (5.4) and (5.5).

Conversely, assume (5.4) and (5.5) hold. It follows from relation (5.4) that

$$a(u_h, v_h - u_h) + \int_\Gamma g\,\lambda_h(v_h - u_h) = l(v_h - u_h) \quad \forall\, v_h \in V_h,$$

which can be rewritten as

$$a(u_h, v_h - u_h) + \int_\Gamma g\,\lambda_h v_h - \int_\Gamma g\,\lambda_h u_h = l(v_h - u_h) \quad \forall\, v_h \in V_h.$$

Then, relation (5.5) implies that

$$a(u_h, v_h - u_h) + \int_\Gamma g\,\lambda_h v_h - \int_\Gamma g\,|u_h| = l(v_h - u_h) \quad \forall\, v_h \in V_h.$$

But since $\lambda_h v_h \le |v_h|$ a.e. on Γ, it follows immediately that u_h is the solution of the discrete problem (5.3). ■

A priori error estimates for the finite element method (5.3) can be found in the literature, e.g. [17, 18]. Here, we are interested in a posteriori analysis for the finite element solution error. By taking $w = u_h$, $q_3^* = -g\,\lambda_h$, and substituting v by $v_h - v$ in (4.6) we obtain

$$\mathcal{R} = \sup_{v \in V} \frac{1}{\|v\|_V} \left\{ \int_\Omega \left[\nabla u_h \cdot \nabla(v - v_h) + u_h(v - v_h) - f\,(v - v_h)\right] + \int_D g\,\lambda_h\,(v - v_h) \right\}. \tag{5.9}$$

Now define $\lambda_{h,D} = \lambda_h \chi_D$. Then the last term of (5.9) can be expressed as $\int_\Omega g\,\lambda_{h,D}(v - v_h)$ or $\int_\Gamma g\,\lambda_{h,D}(v - v_h)$, depending on the nature of the set D. In the following, we derive the a posteriori error estimate for the case $D \subset \Gamma$. For a given $v \in V$, take $v_h = \Pi_h v$ in (5.9). Decompose the integrals into local contributions from each element $K \in \mathcal{P}_h$ and integrate by parts over K to obtain

$$\begin{aligned}
\mathcal{R} &= \sup_{v\in V} \frac{1}{\|v\|_V} \left\{ \int_\Omega \left[\nabla u_h \cdot \nabla(v - \Pi_h v) + u_h(v - \Pi_h v) - f\,(v - \Pi_h v)\right] \right. \\
&\qquad\qquad \left. + \int_\Gamma g\,\lambda_{h,D}(v - \Pi_h v) \right\} \\
&= \sup_{v\in V} \frac{1}{\|v\|_V} \sum_{K\in\mathcal{P}_h} \left\{ \int_{\partial K} \frac{\partial u_h}{\partial n_K}(v - \Pi_h v) + \int_K (-\Delta u_h + u_h - f)(v - \Pi_h v) \right. \\
&\qquad\qquad \left. + \int_{\partial K\cap\Gamma} g\,\lambda_{h,D}(v - \Pi_h v) \right\},
\end{aligned} \tag{5.10}$$

where n_K is the unit outward normal vector to ∂K. Now we define interior residuals for each element $K \in \mathcal{P}_h$ by

$$r_K = -\Delta u_h + u_h - f \quad \text{in } K, \tag{5.11}$$

and side residuals for each side $\gamma \in \partial\mathcal{P}_h$ by

$$R_\gamma = \begin{cases} \left[\frac{\partial u_h}{\partial n}\right] & \text{if } \gamma \in \partial\mathcal{P}_{h,0}, \\ \frac{\partial u_h}{\partial n} + g\,\lambda_{h,D} & \text{if } \gamma \in \partial\mathcal{P}_{h,\Gamma}, \end{cases} \tag{5.12}$$

where the quantity

$$\left[\frac{\partial u_h}{\partial n}\right] = n_K \cdot (\nabla u_h)_K + n_{K'} \cdot (\nabla u_h)_{K'}$$

represents the jump discontinuity in the approximation to the normal derivative on the side γ which separates the neighboring elements K and K'. By using definitions (5.11) and (5.12), relation (5.10) reduces to

$$\mathcal{R} = \sup_{v\in V} \frac{1}{\|v\|_V} \left\{ \sum_{K\in\mathcal{P}_h} \int_K r_K(v - \Pi_h v) + \sum_{\gamma\in\partial\mathcal{P}_h} \int_\gamma R_\gamma(v - \Pi_h v) \right\}. \tag{5.13}$$

Using the estimates (5.1) and (5.2) in (5.13) and applying the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\mathcal{R} &\le \sup_{v\in V} \frac{1}{\|v\|_V} \left\{ \sum_{K\in\mathcal{P}_h} C \left[\|r_K\|_{0;K} h_K + \sum_{\gamma\in\partial K} \|R_\gamma\|_{0;\gamma} h_\gamma^{\frac{1}{2}} \right] |v|_{1;\widetilde{K}} \right\} \\
&\le \sup_{v\in V} \frac{1}{\|v\|_V} \left\{ C\,|v|_{1;\Omega} \Big(\sum_{K\in\mathcal{P}_h} h_K^2 \|r_K\|_{0;K}^2 + \sum_{\gamma\in\partial\mathcal{P}_h} h_\gamma \|R_\gamma\|_{0;\gamma}^2 \Big)^{\frac{1}{2}} \right\} \\
&\le C \left(\sum_{K\in\mathcal{P}_h} h_K^2 \|r_K\|_{0;K}^2 + \sum_{\gamma\in\partial\mathcal{P}_h} h_\gamma \|R_\gamma\|_{0;\gamma}^2 \right)^{\frac{1}{2}}.
\end{aligned} \tag{5.14}$$

In the case $D \subset \Omega$, we can similarly show (5.14) with the interior and side residuals defined by

$$r_K = -\Delta u_h + u_h + g\,\lambda_{h,D} - f \quad \text{in } K, \tag{5.15}$$

$$R_\gamma = \begin{cases} [\frac{\partial u_h}{\partial n}] & \text{if } \gamma \subset \partial K \setminus \Gamma, \\ \frac{\partial u_h}{\partial n} & \text{if } \gamma \subset \partial K \cap \Gamma. \end{cases} \tag{5.16}$$

We summarize the above results in the form of a theorem.

Theorem 5.2 *Let u and u_h be the unique solutions of (2.1) and (5.3), respectively. Then the error $e_h = u - u_h$ satisfies the a posteriori estimate*

$$a(e_h, e_h) = \|\!|e_h|\!\|^2 \le C \left(\sum_{K \in \mathcal{P}_h} h_K^2 \|r_K\|_{0;K}^2 + \sum_{\gamma \in \partial \mathcal{P}_h} h_\gamma \|R_\gamma\|_{0;\gamma}^2 \right) \tag{5.17}$$

where r_K and R_γ are interior and side residuals, respectively, defined by (5.11) and (5.12) in the case $D \subset \Gamma$, and by (5.15) and (5.16) in the case $D \subset \Omega$.

In practical computations, the terms on the right side of (5.17) are regrouped by writing

$$\|\!|e_h|\!\|^2 \le C \sum_{K \in \mathcal{P}_h} \eta_K^2, \tag{5.18}$$

where the local error indicator η_K on each element K, defined by

$$\eta_K^2 = h_K^2 \|r_K\|_{0;K}^2 + \frac{1}{2} h_K \sum_{\gamma \subset \partial K \setminus \Gamma} \|R_\gamma\|_{0;\gamma}^2 + h_K \sum_{\gamma \subset \partial K \cap \Gamma} \|R_\gamma\|_{0;\gamma}^2, \tag{5.19}$$

identifies contributions from each of the elements to the global error.

6 Efficiency of the a posteriori error estimates

This section is devoted to an analysis of the efficiency of the error bound from (5.17) or (5.18). To simplify the presentation we focus on the case $D = \Gamma$. Similar arguments lead to the efficiency proof for the other cases of D.

Our derivation of a lower bound for the error estimator is based on an argument due to Verfürth [42], with special attention paid to the inequality feature of the problem. The argument makes use of the canonical bubble functions constructed for each element $K \in \mathcal{P}_h$ and each side $\gamma \in \partial \mathcal{P}_h$.

Denote by P_K the polynomial space associated with the element K. The following two theorems provide some basic properties of the bubble functions used to derive lower bounds. For more details on bubble functions and proofs see [1].

Theorem 6.1 *Let $K \in \mathcal{P}_h$ and ψ_K be its corresponding bubble function. Then there exists a constant C, independent of h_K, such that for any $v \in P_K$, the following inequalities hold:*

$$C^{-1} \|v\|_{0;K}^2 \le \int_K \psi_K v^2 \le C \|v\|_{0;K}^2,$$
$$C^{-1} \|v\|_{0;K} \le \|\psi_K v\|_{0;K} + h_K |\psi_K v|_{1;K} \le C \|v\|_{0;K}.$$

Theorem 6.2 *Let $K \in \mathcal{P}_h$ and $\gamma \in \partial K$ be one of its sides. Let ψ_γ be the side bubble function corresponding to γ. Then there exists a constant C, independent of h_K, such that for any $v \in P_K$,*

$$C^{-1}\|v\|_{0;\gamma}^2 \le \int_\gamma \psi_\gamma v^2 \le C\|v\|_{0;\gamma}^2,$$
$$h_K^{-1/2}\|\psi_\gamma v\|_{0;K} + h_K^{1/2}|\psi_\gamma v|_{1;K} \le C\|v\|_{0;\gamma}.$$

We now derive a bound for the error estimator. Integrating by parts over each element and using (2.4) and Theorem 5.5 we have, for any $v \in V$,

$$\begin{aligned} a(e_h, v) &= a(u,v) - a(u_h, v) \\ &= \int_\Omega fv - \int_\Gamma g\,\lambda\,v - \int_\Omega (\nabla u_h \cdot \nabla v + u_h v) \\ &= \sum_{K\in\mathcal{P}_h}\int_K (\Delta u_h - u_h + f)v + \sum_{\gamma\in\partial\mathcal{P}_{h,0}}\int_\gamma -\left[\frac{\partial u_h}{\partial n}\right] v + \sum_{\gamma\in\partial\mathcal{P}_{h,\Gamma}}\int_\gamma \left(-\frac{\partial u_h}{\partial n} - g\lambda\right)\, v. \end{aligned}$$

Thus,

$$a(e_h, v) = -\sum_{K\in\mathcal{P}_h}\int_K r_K v - \sum_{\gamma\in\partial\mathcal{P}_h}\int_\gamma R_\gamma v + \sum_{\gamma\in\partial\mathcal{P}_{h,\Gamma}}\int_\gamma g(\lambda_h - \lambda)v, \tag{6.1}$$

where r_K and R_γ are interior and side residuals defined for each element $K \in \mathcal{P}_h$ and each side $\gamma \in \partial\mathcal{P}_h$ by (5.11) and (5.12) respectively. In order to simplify notation, we will omit the subscripts K and γ. Let $\overline{r}$ be a discontinuous piecewise polynomial approximation to the residual r, that is, $\overline{r}|_K \in P_K$. Applying Theorem 6.1 we get

$$\|\overline{r}\|_{0;K}^2 \le C\int_K \psi_K \overline{r}^2. \tag{6.2}$$

Since the function $v = \psi_K\overline{r}$ vanishes on the boundary ∂K, it can be extended to a function in V by 0 to the rest of the domain Ω. Inserting this extended function v in the residual equation (6.1), one obtains

$$a(e_h, \psi_K\overline{r}) = -\int_K r\psi_K\overline{r}.$$

Using this relation, we obtain

$$\int_K \psi_K\overline{r}^2 = \int_K \psi_K\overline{r}(\overline{r} - r) - a(e_h, \psi_K\overline{r}). \tag{6.3}$$

The terms on the right side of (6.3) are bounded by making use of the Cauchy-Schwarz inequality and the second part of Theorem 6.1,

$$\int_K \psi_K\overline{r}^2 \le C\|\overline{r}\|_{0;K}\|\overline{r} - r\|_{0;K} + Ch_K^{-1}|\!|\!|e_h|\!|\!|_K\|\overline{r}\|_{0;K},$$

where $|\!|\!|\cdot|\!|\!|_K$ is the energy norm on the element K. Combined with (6.2) we have

$$\|\overline{r}\|_{0;K} \le C\left(\|\overline{r} - r\|_{0;K} + h_K^{-1}|\!|\!|e_h|\!|\!|_K\right).$$

With the aid of the triangle inequality, finally we get

$$\|r\|_{0;K} \leq \|\overline{r} - r\|_{0;K} + \|\overline{r}\|_{0;K} \leq C \left(\|\overline{r} - r\|_{0;K} + h_K^{-1} \|\!| e_h \|\!|_K \right). \tag{6.4}$$

Consider now a side $\gamma \in \partial\mathcal{P}_{h,0}$. From the first part of Theorem 6.2 it follows that

$$\|R\|_{0;\gamma}^2 \leq C \int_\gamma \psi_\gamma R^2. \tag{6.5}$$

Let $\widetilde{\gamma}$ denote the subdomain of Ω consisting of the side γ and the two neighbouring elements. The function $v = \psi_\gamma R$ vanishes on $\partial\widetilde{\gamma}$ and as before it can be extended continuously to the whole domain Ω by 0 outside $\widetilde{\gamma}$. With this choice of v the residual equation (6.1) reduces to

$$a(e_h, \psi_\gamma R) = -\int_{\widetilde{\gamma}} r\psi_\gamma R - \int_\gamma \psi_\gamma R^2.$$

Therefore,

$$\int_\gamma \psi_\gamma R^2 = -a(e_h, \psi_\gamma R) - \int_{\widetilde{\gamma}} r\psi_\gamma R. \tag{6.6}$$

Applying the Cauchy-Schwarz inequality and the second part of Theorem 6.2 to the terms on the right side of (6.6), we obtain

$$\int_\gamma \psi_\gamma R^2 \leq C\, h_\gamma^{-1/2} \|\!| e_h \|\!|_{\widetilde{\gamma}} \|R\|_{0;\gamma} + C\, h_\gamma^{1/2} \|r\|_{0;\widetilde{\gamma}} \|R\|_{0;\gamma},$$

which, combined with (6.5) and (6.4), implies that for every side $\gamma \in \partial\mathcal{P}_h \setminus \Gamma$,

$$\|R\|_{0;\gamma} \leq C \left(h_\gamma^{-1/2} \|\!| e_h \|\!|_{\widetilde{\gamma}} + h_\gamma^{1/2} \|\overline{r} - r\|_{0;\widetilde{\gamma}} \right). \tag{6.7}$$

Finally, consider those sides γ lying on the boundary Γ. Denote $\overline{R} \in P_K$ an approximation to the residual $R = \frac{\partial u_h}{\partial n} + g\lambda_h$ on γ, $\gamma \subset \partial K$. The first part of Theorem 6.2 implies

$$\|\overline{R}\|_{0;\gamma}^2 \leq C \int_\gamma \psi_\gamma \overline{R}^2. \tag{6.8}$$

Define the function $v = \psi_\gamma \overline{R}$ and let $\widetilde{\gamma}$ be the element whose boundary contains the side γ. Then $v|_{\partial\widetilde{\gamma}\setminus\gamma} = 0$. Extend this function to the whole domain by zero value outside $\widetilde{\gamma}$. The residual equation (6.1), with this choice of v, becomes

$$a(e_h, \psi_\gamma \overline{R}) = -\int_{\widetilde{\gamma}} r\psi_\gamma \overline{R} - \int_\gamma R\psi_\gamma \overline{R} + \int_\gamma g(\lambda_h - \lambda)\psi_\gamma \overline{R}$$

which leads to

$$\int_\gamma \psi_\gamma \overline{R}^2 = \int_\gamma \psi_\gamma \overline{R}(\overline{R} - R) - a(e_h, \psi_\gamma \overline{R}) - \int_{\widetilde{\gamma}} r\psi_\gamma \overline{R} + \int_\gamma g(\lambda_h - \lambda)\psi_\gamma \overline{R}. \tag{6.9}$$

As before, the first three terms on the right side of (6.9) can be bounded by applying Theorem 6.2 and the Cauchy-Schwarz inequality. Using (6.8), we then obtain for each side $\gamma \in \partial\mathcal{P}_{h,\Gamma}$,

$$\begin{aligned}\|\overline{R}\|_{0;\gamma}^2 \leq & C\left(\|\overline{R}\|_{0;\gamma}\|\overline{R}-R\|_{0;\gamma} + h_\gamma^{-1/2}\|\overline{R}\|_{0;\gamma}|\!|\!| e_h |\!|\!|_{\widetilde{\gamma}} + h_\gamma^{1/2}\|\overline{R}\|_{0;\gamma}\|r\|_{0;\widetilde{\gamma}}\right) \\ & + \int_\gamma g(\lambda_h - \lambda)\psi_\gamma \overline{R}.\end{aligned}$$

Multiplying this inequality by h_γ and summing over all sides $\gamma \in \partial\mathcal{P}_{h,\Gamma}$ we get

$$\begin{aligned}\sum_{\gamma\in\partial\mathcal{P}_{h,\Gamma}} h_\gamma \|\overline{R_\gamma}\|_{0;\gamma}^2 \leq C & \sum_{\gamma\in\partial\mathcal{P}_{h,\Gamma}} h_\gamma^{1/2}\|\overline{R_\gamma}\|_{0;\gamma} h_\gamma^{1/2}\|\overline{R_\gamma}-R_\gamma\|_{0;\gamma} \\ & + C \sum_{\gamma\in\partial\mathcal{P}_{h,\Gamma}} h_\gamma^{1/2}\|\overline{R_\gamma}\|_{0;\gamma}|\!|\!| e_h |\!|\!|_{\widetilde{\gamma}} \\ & + C \sum_{\gamma\in\partial\mathcal{P}_{h,\Gamma}} h_\gamma^{1/2}\|\overline{R_\gamma}\|_{0;\gamma} h_\gamma \|r_{\widetilde{\gamma}}\|_{0;\widetilde{\gamma}} + |R_{h,\Gamma}|, \end{aligned} \tag{6.10}$$

where

$$R_{h,\Gamma} = \sum_{\gamma\in\partial\mathcal{P}_{h,\Gamma}} \int_\gamma g\left(\lambda - \lambda_h\right) h_\gamma \psi_\gamma \overline{R_\gamma}.$$

We can bound $R_{h,\Gamma}$ as follows,

$$\begin{aligned}|R_{h,\Gamma}| & \leq \sum_{\gamma\in\partial\mathcal{P}_{h,\Gamma}} h_\gamma^{1/2}\|g\left(\lambda-\lambda_h\right)\|_{0;\gamma} h_\gamma^{1/2}\|\psi_\gamma\overline{R_\gamma}\|_{0;\gamma} \\ & \leq C\left(\sum_{\gamma\in\partial\mathcal{P}_{h,\Gamma}} h_\gamma\|\lambda-\lambda_h\|_{0;\gamma}^2\right)^{1/2}\left(\sum_{\gamma\in\partial\mathcal{P}_{h,\Gamma}} h_\gamma\|\overline{R_\gamma}\|_{0;\gamma}^2\right)^{1/2}.\end{aligned}$$

Use this bound in (6.10) and apply the Cauchy-Schwarz inequality to get

$$\begin{aligned}\sum_{\gamma\in\partial\mathcal{P}_{h,\Gamma}} h_\gamma\|\overline{R_\gamma}\|_{0;\gamma}^2 \leq C & \left(\sum_{\gamma\in\partial\mathcal{P}_{h,\Gamma}} h_\gamma\|\overline{R_\gamma}-R_\gamma\|_{0;\gamma}^2 + \sum_{\gamma\in\partial\mathcal{P}_{h,\Gamma}} h_\gamma^2\|r_{\widetilde{\gamma}}\|_{0;\widetilde{\gamma}}^2\right. \\ & \left. + |\!|\!| e_h |\!|\!|^2 + \sum_{\gamma\in\partial\mathcal{P}_{h,\Gamma}} h_\gamma\|\lambda-\lambda_h\|_{0;\gamma}^2\right).\end{aligned} \tag{6.11}$$

Combining (6.4) and (6.11), we finally conclude that

$$\begin{aligned}\sum_{\gamma\in\partial\mathcal{P}_{h,\Gamma}} h_\gamma\|R_\gamma\|_{0;\gamma}^2 \leq C & \left(\sum_{\gamma\in\partial\mathcal{P}_{h,\Gamma}} h_\gamma\|\overline{R_\gamma}-R_\gamma\|_{0;\gamma}^2 + \sum_{\gamma\in\partial\mathcal{P}_{h,\Gamma}} h_\gamma^2\|\overline{r_{\widetilde{\gamma}}}-r_{\widetilde{\gamma}}\|_{0;\widetilde{\gamma}}^2\right. \\ & \left. + |\!|\!| e_h |\!|\!|^2 + \sum_{\gamma\in\partial\mathcal{P}_{h,\Gamma}} h_\gamma\|\lambda-\lambda_h\|_{0;\gamma}^2\right).\end{aligned} \tag{6.12}$$

Summarizing (6.4), (6.7) and (6.12), we thus have the following efficiency result.

Theorem 6.3 *Let η_K be defined as in (5.19). Then we have*

$$\sum_{K\in\mathcal{P}_h}\eta_K^2 \le C\left(|||e_h|||^2 + \sum_{\gamma\in\partial\mathcal{P}_{h,\Gamma}} h_\gamma\|\lambda-\lambda_h\|_{0;\gamma}^2 \right.$$
$$\left. + \sum_{K\in\mathcal{P}_h} h_K^2\|r_K-\overline{r_K}\|_{0,K}^2 + \sum_{\gamma\in\partial\mathcal{P}_{h,\Gamma}} h_\gamma\|R_\gamma-\overline{R_\gamma}\|_{0,\gamma}^2\right) \tag{6.13}$$

with discontinuous piecewise polynomial approximations $\overline{r_K}$, $\overline{R_\gamma}$ of r_K, R_γ.

We notice that due to the inequality nature of the variational problem, in the efficiency bound (6.13) of the error estimator, there is an extra term. A sharp bound of this extra term is currently an open problem. Nevertheless, in the next section, we will present numerical results showing that the presence of this term in (6.13) does not have an effect on the efficiency of the error estimator.

7 Numerical examples

We present several numerical examples to show the effectiveness of the error estimator (5.17), one example in one dimension and another in two dimensions. Linear and bilinear elements are used in the experiments. The discretized solution is computed by solving the equivalent minimization problem using an over-relaxation method with a relative error tolerance, in the maximum norm, of 10^{-6} (see [17], [18]).

In order to show the effectiveness of the adaptive procedure we compare numerical convergence orders of the approximate solutions. We compute these orders by considering families of uniform and adaptively refined partitions. Consider a sequence of finite element solutions u_h^{un} based on uniform partitions of the domain Ω. Starting with an initial coarse partition $\mathcal{P}_1$, we construct a family of nested meshes by subdividing each element into two congruent elements for one-dimensional case and into four congruent elements for two-dimensional case. The solution from the most refined mesh will be taken as the "exact" solution u that will be used to compute the errors of the approximate solutions obtained on the other meshes.

Adaptive finite element solutions are obtained by the following algorithm:

1. Start with the initial partition $\mathcal{P}_h$ and corresponding finite element subspace V_h.

2. Compute the finite element solution $u_h^{ad}\in V_h$.

3. For each element $K\in\mathcal{P}_h$ compute the error estimator η_K defined in (5.19).

4. Let $\eta = \frac{1}{N}\sum_{K\in\mathcal{P}_h}\eta_K$ with N being the number of elements in partition $\mathcal{P}_h$. An element K is marked for refinement if $\eta_K > \mu\,\eta$, where μ is a prescribed threshold. In our examples, $\mu = 0.5$.

5. Perform refinement and obtain a new triangulation $\mathcal{P}_h$.

6. Return to step 2.

In the computation of the error indicator η_K we make use of the multiplier λ_h defined on $D \subset \Gamma$. In what follows we describe how λ_h can be (approximately) recovered from the solution u_h using characterization (5.4).

Denote by $\{\boldsymbol{x}^i\}_{i=1}^m$ the nodes of the partition $\mathcal{P}_h$ belonging to $\overline{D}$, and let $\{\phi_i\}_{i=1}^m$ be the basis functions corresponding to the nodes $\{\boldsymbol{x}^i\}$. We first determine a piecewise linear function

$$\lambda_{h,0} = \sum_{i=1}^m \lambda_{h,0}^i \phi_i$$

by requiring an analogue of (5.4):

$$a(u_h, v_h) + \int_D g\, \lambda_{h,0} v_h = l(v_h) \quad \forall\, v_h \in V_h. \tag{7.1}$$

Denote $\boldsymbol{\lambda}_{h,0} = (\lambda_{h,0}^1, \dots, \lambda_{h,0}^m)^T$. We then project the components of $\boldsymbol{\lambda}_{h,0}$ onto the interval $[-1, 1]$ to get $\boldsymbol{\lambda}_{h,1} = (\lambda_{h,1}^1, \dots, \lambda_{h,1}^m)^T$. The piecewise linear approximation of the multiplier λ_h on $D \subset \Gamma$ can be computed as

$$\lambda_{h,1} = \sum_{i=1}^m \lambda_{h,1}^i \phi_i. \tag{7.2}$$

We briefly comment on the method for finding $\lambda_{h,0}$. Let $n = \dim V_h$. Denote by $\boldsymbol{K}$ the standard $(n \times n)$ stiffness matrix and by $\boldsymbol{l} \in \mathbb{R}^n$ the standard load vector. Let $\boldsymbol{u} \in \mathbb{R}^n$ be the nodal value vector of the finite element solution u_h. Then the algebraic representation of (7.1) becomes

$$(\boldsymbol{K}\boldsymbol{u}, \boldsymbol{v})_{\mathbb{R}^n} + (\boldsymbol{M}\boldsymbol{\lambda}_{h,0}, \boldsymbol{v}_c)_{\mathbb{R}^m} = (\boldsymbol{l}, \boldsymbol{v})_{\mathbb{R}^n} \quad \forall\, \boldsymbol{v} \in \mathbb{R}^n, \tag{7.3}$$

where $\boldsymbol{v}_c$ denotes the subvector of $\boldsymbol{v}$, containing the nodal values of v_h at the nodes $\{\boldsymbol{x}^i\}_{i=1}^m \subset \overline{D}$ and $\boldsymbol{M}$ is a tridiagonal $(m \times m)$ matrix. We can write $\boldsymbol{v} = (\boldsymbol{v}_i^T, \boldsymbol{v}_c^T)^T \in \mathbb{R}^{n-m} \times \mathbb{R}^m$ by assuming that the components of $\boldsymbol{v}_c$ are listed last. We similarly split $\boldsymbol{l}$ to $\boldsymbol{l}_i$ and $\boldsymbol{l}_c$. This decomposition yields a block structure for $\boldsymbol{K}$,

$$\boldsymbol{K} = \begin{pmatrix} \boldsymbol{K}_{ii} & \boldsymbol{K}_{ic} \\ \boldsymbol{K}_{ci} & \boldsymbol{K}_{cc} \end{pmatrix}.$$

Then (7.3) is equivalent to the following two relations:

$$\begin{aligned} \boldsymbol{K}_{ii}\boldsymbol{u}_i + \boldsymbol{K}_{ic}\boldsymbol{u}_c &= \boldsymbol{l}_i, \\ \boldsymbol{K}_{ci}\boldsymbol{u}_i + \boldsymbol{K}_{cc}\boldsymbol{u}_c + \boldsymbol{M}\boldsymbol{\lambda}_{h,0} &= \boldsymbol{l}_c. \end{aligned}$$

Once the approximate solution u_h is computed, we can obtain from the second relation that

$$\boldsymbol{\lambda}_{h,0} = \boldsymbol{M}^{-1}(\boldsymbol{l}_c - \boldsymbol{K}_{ci}\boldsymbol{u}_i - \boldsymbol{K}_{cc}\boldsymbol{u}_c).$$

In the numerical examples below, we use u_h^{un} for finite element solutions on uniform meshes, and u_h^{ad} for finite element solutions on adaptive meshes. Since adaptive solutions are involved, numerical solution errors will be plotted against the number of degrees of freedom, rather than the meshsize.

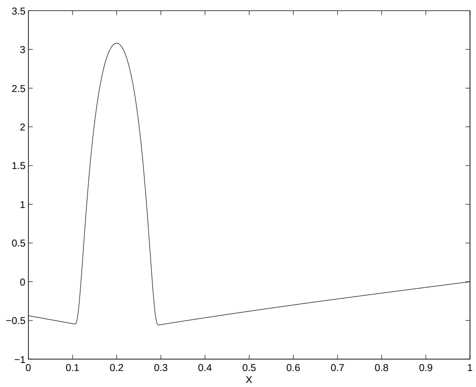

Figure 1: Example 7.1. Solution corresponding to $h = 2^{-12}$.

Example 7.1 Let $\Omega = [0,1]$ and $D = \Gamma \equiv \{0,1\}$. The model problem (2.1) becomes: Find $u \in H^1(0,1)$ such that

$$\int_0^1 \Big(u'(v-u)' + u(v-u)\Big)\,dx + g\Big(|v(0)| + |v(1)| - |u(0)| - |u(1)|\Big) \geq \int_0^1 f(v-u)\,dx \quad \forall\, v \in H^1(0,1). \tag{7.4}$$

We choose the right side function

$$f(x) = \begin{cases} \frac{1}{\varepsilon}\left(-\frac{1}{\varepsilon^2}\frac{6y^4-2}{(y^2-1)^4} + 1\right) e^{\frac{1}{y^2-1}} - \frac{1}{\varepsilon} e^{\frac{4\varepsilon^2}{1-4\varepsilon^2}} & \text{if } |y| < 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $y = \frac{x-x_0}{\varepsilon}$, $0 < x_0 < 1$, and $\varepsilon > 0$ is a parameter used to control the roughness of the solution. In this example, we take $g = 1$, $x_0 = 0.2$ and $\varepsilon = 0.1$. The solution of the problem (7.4) is shown in Figure 1. For the computation of errors, the "exact" solution u of the problem (7.4) is approximated by extrapolation of solutions obtained on uniform partitions with mesh sizes as small as $h = 2^{-12}$. A sequence of approximate solutions u_h^{ad} is obtained by using an adaptive strategy based on estimate (5.18). It should be mentioned that in one-dimensional case, the error indicators η_K do not include terms with side residuals R_γ and thus there is no need to compute multipliers λ_h. The errors $\|u - u_h^{un}\|_{1;\Omega}$ for finite element solutions on uniform meshes and $\|u - u_h^{ad}\|_{1;\Omega}$ for adaptive finite element solutions are compared in Figure 2. We observe a substantial improvement in the efficiency of the adaptive solution as compared to the uniform mesh solution; e.g., the error of the adaptive solution (7.487×10^{-2}) with 326 nodes is comparable to that of the uniform mesh solution (6.623×10^{-2}) with 4097 nodes.

Example 7.2 Let $\Omega = [0,1] \times [0,1]$ and $D = \Gamma$. The problem solved is: Find $u \in H^1(\Omega)$ such that

$$\int_\Omega [\nabla u \cdot \nabla (v-u) + u\,(v-u)]\,dx + g\int_\Gamma |v|\,ds - g\int_\Gamma |u|\,ds \geq \int_\Omega f\,(v-u)\,dx \quad \forall\, v \in H^1(\Omega),$$

where

$$f = -\Delta w + w,$$
$$w = w_1 - w_2,$$

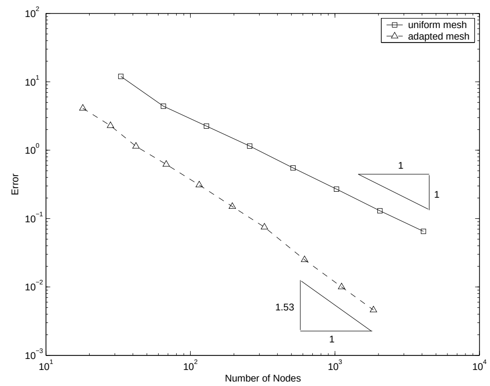

Figure 2: Example 7.1. $\|u - u_h^{un}\|_{1;\Omega}$ (□) vs. $\|u - u_h^{ad}\|_{1;\Omega}$ (△).

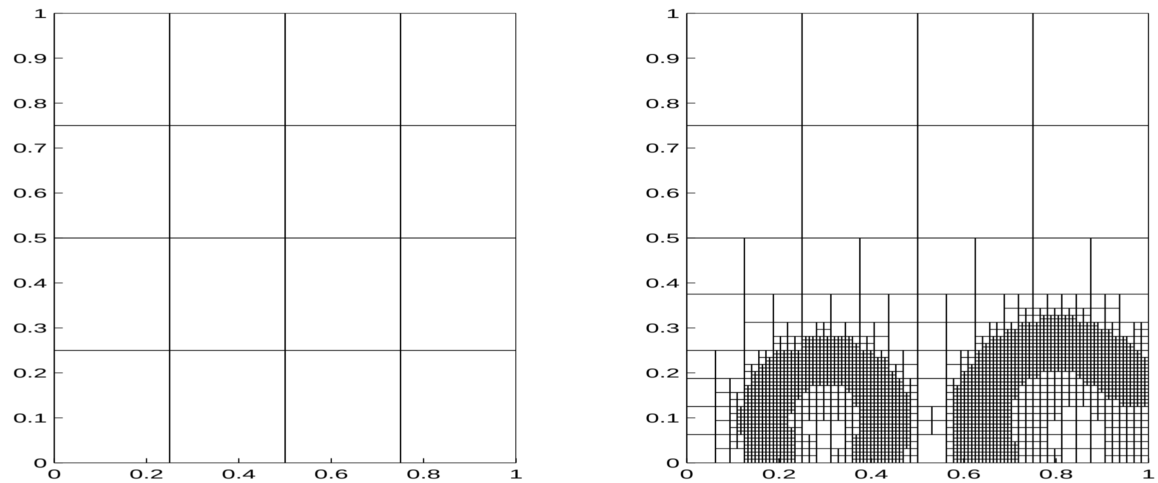

Figure 3: Example 7.2. Initial partition and locally refined partition after 5 iterations.

and for $i = 1, 2$,

$$w_i(\boldsymbol{x}) = \begin{cases} \exp(1/(r_i^2 - 1)) & \text{if } r_i < 1, \\ 0 & \text{otherwise} \end{cases}$$

with $r_i = [(x_1 - x_{1,0}^{(i)})^2 + (x_2 - x_{2,0}^{(i)})^2]^{1/2}/\varepsilon_i$.

In the experiments we let $g = 1$, $x_{1,0}^{(1)} = 0.8$, $x_{2,0}^{(1)} = 0.1$, $x_{1,0}^{(2)} = 0.3$, $x_{2,0}^{(2)} = 0.1$, $\varepsilon_1 = 0.25$, $\varepsilon_2 = 0.2$. We consider two types of partitioning. First, we use rectangular partitioning and the corresponding continuous piecewise bilinear functions. We start with an initial partition consisting of 16 square elements with side length $h = 1/4$ shown on the left plot in Figure 3. The numerical solution corresponding to $h = 1/256$, shown in Figure 4, is taken as the "exact" solution u.

We use the adaptive strategy to obtain a sequence of approximate solutions u_h^{ad}. An element marked for refinement is divided into four elements by joining the modpoints of edges. This refinement technique introduces hanging nodes, that is, midpoints of edges where not all of the neighbouring elements have been divided. The values of the function at the hanging nodes are fixed to be the linear interpolants of the unknowns corresponding to neighbouring regular nodes and therefore, the conformity $V_h \subset V$ is

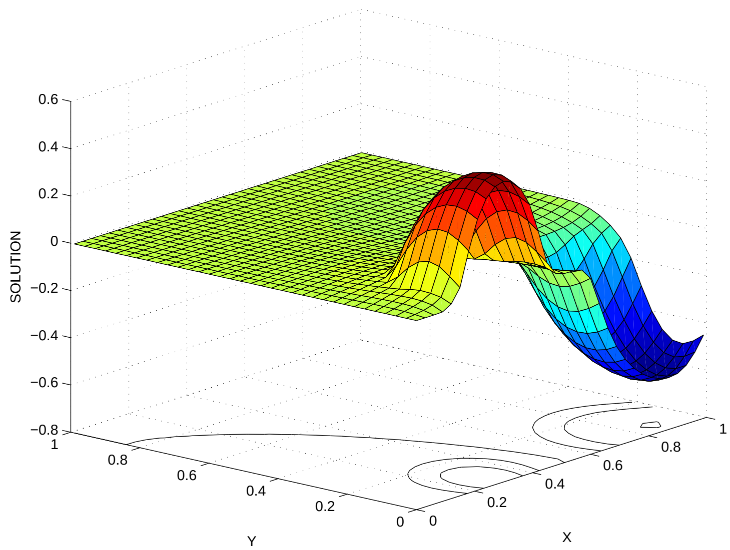

Figure 4: Example 7.2. Solution on a uniform rectangular mesh with 66,049 nodes.

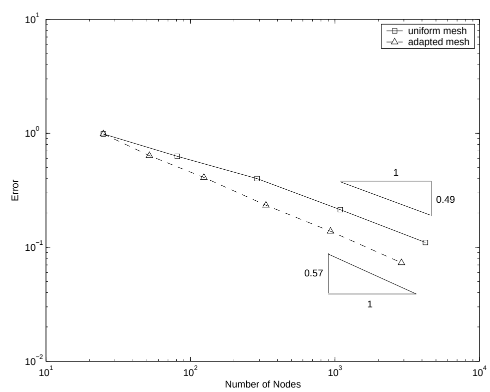

Figure 5: Example 7.2. Square partition. $\|u - u_h^{un}\|_{1;\Omega}$ ($\square$) vs. $\|u - u_h^{ad}\|_{1;\Omega}$ ($\triangle$).

preserved. Such a finite element mesh after 5 iterations is shown on the right plot in Figure 3. In Figure 5 we plot the error values $\|u - u_h^{un}\|_{1;\Omega}$ and $\|u - u_h^{ad}\|_{1;\Omega}$ versus the number of degrees of freedom. Again, we observe a substantial improvement of the efficiency using adaptively refined meshes.

To have an idea of the convergence behaviour of the discrete Lagrange multipliers, we analyze the errors $\|\lambda - \lambda_h\|_{0;\Gamma}$ corresponding to the sequence of uniform refinements. Here, λ is the Lagrange multiplier corresponding to the parameter $h = 1/256$. Figure 6 provides the error values $\|u - u_h^{un}\|_{1;\Omega}$ and $h^{1/2}\|\lambda - \lambda_h\|_{0;\Gamma}$. The numerical convergence order of $h^{1/2}\|\lambda - \lambda_h\|_{0;\Gamma}$ is obviously higher than that of $\|u - u_h^{un}\|_{1;\Omega}$, indicating that the second term in the efficiency bound (6.3) is expected to be of higher order compared to the first term $\|e_h\|_{1;\Omega}^2$. Graphs of $\lambda_{h,0}$ and $\lambda_{h,1}$ with $h = 1/256$ are provided in Figures 7 and 8.

Next we consider a triangular partitioning and the continuous piecewise linear finite elements. We start with a coarse triangulation shown on the left plot in Figure 9. Here the interval $[0, 1]$ is divided

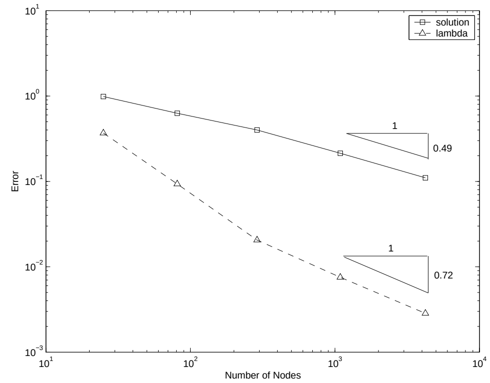

Figure 6: Example 7.2. Uniform square partition. $\|u - u_h^{un}\|_{1;\Omega}$ ($\square$) vs. $h^{1/2}\|\lambda - \lambda_h\|_{0;\Gamma}$ ($\triangle$).

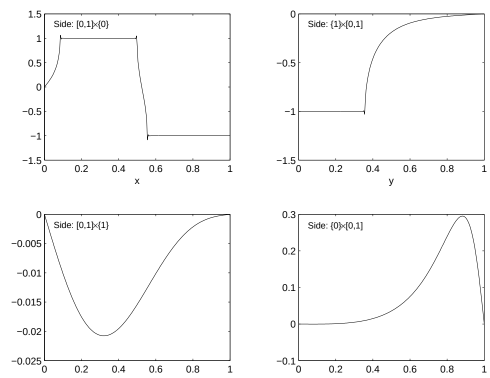

Figure 7: Example 7.2. Plots of $\lambda_{h,0}$.

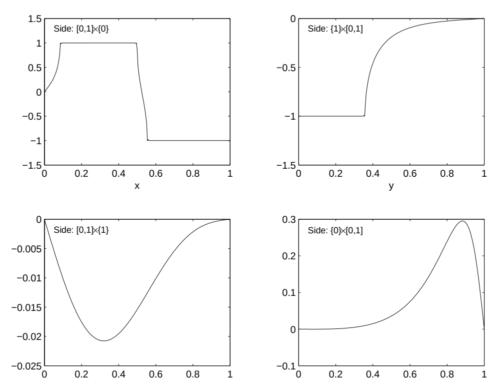

Figure 8: Example 7.2. Plots of $\lambda_{h,1}$.

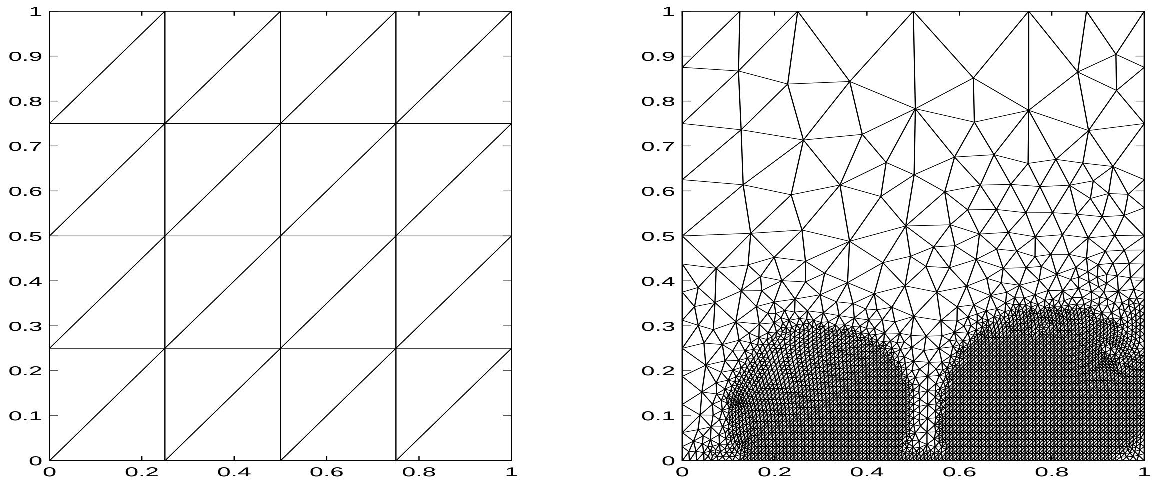

Figure 9: Example 7.2. Initial partition and locally refined partition after 5 iterations.

into $1/h$ equal parts with $h = 1/4$ which is successively halved. The numerical solution corresponding to $h = 1/256$ is taken as the “exact” solution u.

We use the regular refinement technique where the triangle is divided into four triangles by joining the midpoints of edges. For a detailed description of this and other refinement techniques currently used see e.g. [42] and references therein. Also, in order to improve the quality of triangulation, a smoothing procedure is used after each refinement. For each triangle K of the triangulation we compute the triangle quality measure defined by

$$Q(K) = \frac{4\sqrt{3}\,\text{area}(K)}{h_1^2 + h_2^2 + h_3^2},$$

where h_i, $i = 1, 2, 3$, are the side lengths of the triangle K. Note that $Q(K) = 1$ if $h_1 = h_2 = h_3$. A triangle is viewed to be of acceptable quality if $Q > 0.6$, otherwise we modify the mesh by moving the interior nodes toward the center of mass of the polygon formed by the adjacent triangles.

A finite element mesh obtained after 5 consecutive adaptive iterations is shown in Figure 9. Errors $\|u - u_h^{un}\|_{1;\Omega}$ and $\|u - u_h^{ad}\|_{1;\Omega}$ are provided in Figure 10. Figure 11 shows the comparison of $\|u - u_h^{un}\|_{1;\Omega}$ and $h^{1/2}\|\lambda - \lambda_h\|_{0;\Gamma}$ on uniformly refined meshes. Again, we observe improved efficiency with the use of the adaptive refinement. Moreover, in (6.13), the second term within the parentheses is of higher order than the first term there.

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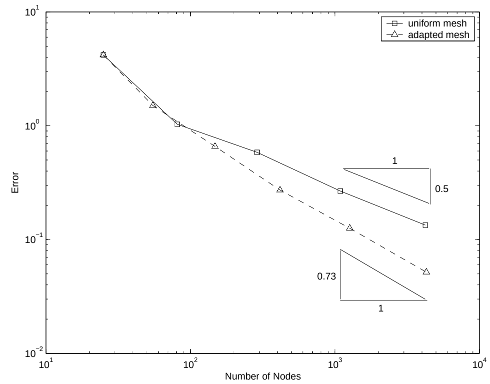

Figure 10: Example 7.2. Triangular partition. $\|u - u_h^{un}\|_{1;\Omega}$ ($\square$) vs. $\|u - u_h^{ad}\|_{1;\Omega}$ ($\triangle$).

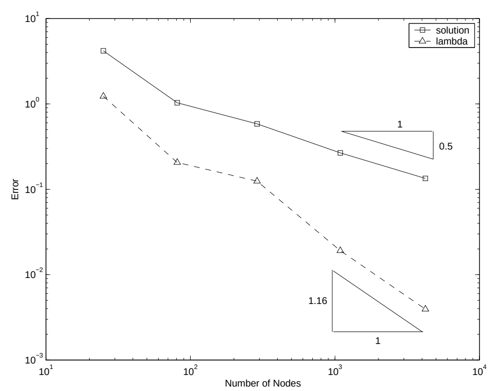

Figure 11: Example 7.2. Uniform triangular partition. $\|u - u_h^{un}\|_{1;\Omega}$ ($\square$) vs. $h^{1/2}\|\lambda - \lambda_h\|_{0;\Gamma}$ ($\triangle$).

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